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MECHANICS

An Unsteady Pseudoshock Model for Barotropic Gas Flow

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Abstract—A mathematical unsteady pseudoshock model describing the continuous transition from supersonic to subsonic flow is constructed for a barotropic gas flow in a long flat channel or a nozzle. The model is based on a two-layer scheme of flow with mass transfer including a potential supersonic core and a turbulent boundary layer.

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1. A complex gas-dynamic structure called pseudoshock is implemented in long channels under supersonic-flow drag [1]. In supersonic gas flow, unsteady processes affecting the flow structure develop at the interaction with the boundary layer. As follows from analysis of numerous experimental investigations of processes of supersonic-flow drag, which were carried out over several dozen years, the phenomena under investigation are characterized by different modes, scales, etc. Reviews of the theoretical and experimental results are given in [2, 3]. The mathematical models of pseudoshock assume, as a rule, a steady or quasi-steady mode of flow in the region of transition from supersonic to subsonic flow due to the development of a turbulent boundary layer. In this case, the problem remains open concerning what conditions control the shock position in the downstream flow and how the mechanism of transfer of perturbations for upstream flows with a supersonic "core" is implemented. The experiments show that the introduction of various perturbations into the flow behind a pseudoshock (narrowing of the channel, inflow of mass or energy, and chemical reactions) results in upstream displacement of the pseudoshock. The periodic mechanical or energy effect on the flow can cause forced vibrations of the pseudoshock around the new quasi-steady position [4].

In this study, we propose a model enabling us possibly to determine the qualitative and quantitative characteristics of the forced vibrations of the pseudoshock under the periodic effect on the flow at the outlet portion of the channel. Within the framework of the two-layer scheme of flow (the potential supersonic core and the turbulent boundary layer), we constructed a mathematical model of an unsteady pseudoshock in the flat channel describing a continuous transition from supersonic to subsonic flow. The equations of motion are represented as the set of five laws of conservation with the right-hand side. The structure of steady solutions was analyzed, and the flows in front of an obstacle were found. On the basis of numerical modeling, we showed the evolution of the pseudoshock under a periodic change in the channel outlet cross section. Verification of the model by comparison with the experimental data is carried out.

2. We consider a plane-parallel flow of barotropic gas in which a pseudoshock is implemented. One of the versions of the control of downstream conditions is a local obstacle in the channel providing the transonic mode of the flow in its vicinity, for example, the Laval nozzle. In the pseudoshock, the mechanism of transition from supersonic to subsonic flow is related to the development of the turbulent boundary layer, the average flow velocity in which is less than that of the mainstream motion. If we consider that the boundary layer is formed in the vicinity of the channel walls, and the flow in the flat long channel is symmetric with respect to its axes, it suffices to consider the two-layer scheme of the flow. Under the specified assumptions, the one-dimensional equations of motion of the barotropic gas take the form

$$
\begin{gathered}
(\rho h)_t + (u\rho h)_x = -\sigma q \rho, \quad (\rho\eta)_t + (v\rho\eta)_x = \sigma q \rho, \\
(u\rho h + v\rho\eta)_t + (u^2\rho h + (v^2 + q^2)\rho\eta + (h+\eta)p)_x \\
= -p z_x - c_f \rho v^2, \quad u_t + \left(\frac{u^2}{2} + i\right)_x = 0, \\
(u^2\rho h + (v^2 + q^2)\rho\eta + 2(h+\eta)\rho\varepsilon)_t \\
+ (u^3\rho h + (v^2 + 3q^2)v\rho\eta \\
+ 2(\varepsilon\rho + p)(uh + v\eta))_x = 2p z_t - \sigma\kappa\rho q^3 - 2c_f\rho v^3.
\end{gathered}
\tag{1}
$$

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Here, ρ is the gas density, h and η are the thicknesses of the potential and turbulent layers, respectively; u and v are the velocity of gas in these layers; and the value of q characterizes the shift of velocity in the turbulent layer. The constants σ, κ, and c_f are the empirical parameters responsible for the intensity of the mass transfer (the rate of involving the gas into the turbulent layer), the dissipation of energy, and the friction at the channel walls. The half-width of the channel is equal to $h + \eta = H_0 - z$, where the function $z = z(t, x)$ sets the relative contraction of the channel. The pressure, speed of sound, internal energy, and enthalpy of the polytrophic gas are set by the formulas

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad c^2 = \frac{\gamma p}{\rho}, \quad \varepsilon = \frac{p}{(\gamma-1)\rho},$$
$$i = \frac{c^2}{\gamma - 1}.$$

In the dimensionless variables $H_0 = 1$, $\rho_0 = 1$, $p_0 = \frac{c_0^2\rho_0}{\gamma}$, and $c_0^2 = 1$.

Similar two-layer and three-layer models were already applied for describing turbulent bores and the mixture layers in shear flows of a fluid with a free boundary [5, 6].

2.1. The derivation of set (1) is based on averaging the two-dimensional equations of motion of gas over the channel cross section in the approximation of the long-wave theory [5]:

$$\begin{gathered}\rho_t + (U\rho)_x + (V\rho)_y = 0,\\ (U\rho)_t + (U^2\rho + p)_x + (UV\rho)_y = 0, \quad p_y = 0, \\ e_t + ((e+p)U)_x + ((e+p)V)_y = 0\end{gathered} \tag{2}$$

with the boundary conditions on the axis of symmetry and the upper wall of the channel

$$V|_{y=0} = 0, \quad z_t + Uz_x + V|_{y = H_0 - z} = 0. \tag{3}$$

Here, U and V are the components of the gas-velocity vector, while $e = \left(\varepsilon + \frac{U^2}{2}\right)\rho$ is the total energy. In the case of a barotropic gas, the last equation in set (2) is a consequence of the previous relations.

We assume that the flow is potential; i.e., $y \in (0, h)$ in the layer $U = u(t, x)$. For the averaged description of the flow in the turbulent boundary layer $y \in (h, H_0 - z)$, we use the average velocity of the flow $v(t, x)$ and its root-mean-square deviation $q^2(t, x)$ determined by the formulas

$$v = \frac{1}{\eta}\int_h^{H_0 - z} U dy, \quad q^2 = \frac{1}{\eta}\int_h^{H_0 - z}(U - v)^2 dy.$$

With taking into account the identity $U = v + (U - v)$, we calculate the integrals

$$\begin{gathered}\int_h^{H_0 - z} U^2 dy = (v^2 + q^2)\eta,\\ \int_h^{H_0 - z} U^3 dy = (v^2 + 3q^2)v\eta + P.\end{gathered} \tag{4}$$

According to [7], the correlation P is small and can be rejected if the initial data for set (2) satisfy the condition of weak vorticity $U_y = O(\delta^{\beta})$, $0 < \beta < 1$ ($\delta \ll 1$ is the ratio of the characteristic vertical channel scale H_0 and the horizontal scale L_0). Further, we assume that this condition is fulfilled.

The integration of the first equation of set (2) over the variable y from zero to $H_0 - z$ and the use of boundary conditions (3) give the law of conservation of the gas mass in the channel

$$(\rho h + \rho\eta)_t + (u\rho h + v\rho\eta)_x = 0.$$

Following [5], we assume that the rate of involving the gas from the potential core into the turbulent boundary layer is proportional to $q\rho$. Then the balance relation of the gas mass in the layers accepts the form of the first two equations of set (1). The third and fourth (at $c_f = 0$) equations of set (1) arise from averaging the second equation in set (2) over the potential-layer thickness and the height of the entire channel. When calculating the integrals, we used boundary conditions (3) and formulas (4). The averaging of the law of conservation of energy (last equation (2)) over the channel height gives the closure of the fifth equation of set (1) in the case of $c_f = 0$ and $\kappa = 0$.

When modeling real flows, it is expedient to take into account the friction and energy dissipation. For this reason, we added the corresponding terms containing the empirical parameters c_f and κ in the right-hand side of balance relations (1).

For finding the characteristics of set (1) of equations and constructing the steady solutions, it is convenient to use the consequences:

$$\begin{gathered}v_t + vv_x + \frac{c^2}{\rho}\rho_x + \frac{1}{\rho\eta}(\rho\eta q^2)_x = \frac{\sigma q}{\eta}(u - v) - \frac{c_f v^2}{\eta},\\ q_t + (vq)_x = \frac{\sigma}{2\eta}((u - v)^2 - (1 + \kappa)q^2).\end{gathered}$$

In the general case, set (1) is not hyperbolic. However, the presence of at least three material characteristics, including one contact and two sonic, allows us to use laws of conservation (1) for constructing the conservative numerical schemes.

2.2. During the evolution of flow, the turbulent boundary layer is extended and, when the axis of symmetry $y = 0$ is achieved, two-layer model (1) passes into the single-layer model ($h = 0$). In this case, the equations of motion take the form

$$
\begin{gathered}
(\rho\eta)_t + (v\rho\eta)_x = 0,\\
(v\rho\eta)_t + ((v^2+q^2)\rho\eta + \eta p)_x = -pz_x - c_f\rho v^2,\\
((v^2+q^2+2\varepsilon)\rho\eta)_t \\
+ ((v^2+3q^2+2\varepsilon)v\rho\eta + 2pv\eta)_x \\
= 2pz_t - \sigma\kappa\rho q^3 - 2c_f\rho v^3 \quad (\eta = H_0 - z).
\end{gathered} \tag{5}
$$

Equations (5) of the single-layer flow are hyperbolic and have one contact $\frac{dx}{dt} = v$ and two sonic characteristics

$$\frac{dx}{dt} = v \pm \sqrt{c^2 + 3q^2},$$

determining the velocities of propagation of perturbations in the gas.

3. The plane-parallel steady flows of barotropic gas within the framework of the assumptions made are determined from the solution of the equations (the stroke means differentiation with respect to x)

$$
\begin{gathered}
(u\rho h)' = -\sigma q\rho, \quad (v\rho\eta)' = \sigma q\rho, \quad uu' + \frac{c^2}{\rho}\rho' = 0,\\
vv' + \frac{c^2}{\rho}\rho' + \frac{1}{\rho\eta}(\rho\eta q^2)' = \frac{\sigma q}{\eta}(u - v) - \frac{c_f v^2}{\eta},\\
(vq)' = \frac{\sigma}{2\eta}((u-v)^2 - (1+\kappa)q^2).
\end{gathered} \tag{6}
$$

Set (6) can be written in the form resolved with respect to the derivatives

$$
\begin{gathered}
\rho' = \frac{\sigma q\rho}{\Delta}\left(\frac{2v - u + F}{v^2 - 3q^2} - \frac{1}{u} + \frac{\eta z'}{\sigma q}\right), \quad u' = -\frac{c^2}{\rho u}\rho',\\
\eta' = \frac{\sigma q}{u} + \left(1 - \frac{c^2}{u^2}\right)\frac{h}{\rho}\rho' - z', \quad v' = \frac{\sigma q}{\eta} - \frac{v}{\rho}\rho' - \frac{v}{\eta}\eta',\\
q' = -\frac{q}{v}v' + \frac{\sigma}{2\eta v}((u-v)^2 - (1+\kappa)q^2),
\end{gathered} \tag{7}
$$

where

$$
\begin{gathered}
\Delta = H_0 - z - \frac{hc^2}{u^2} - \frac{\eta c^2}{v^2 - 3q^2},\\
F = \frac{(u-v)^2 - (3+\kappa)q^2}{v} + \frac{c_f v^2}{\sigma q}.
\end{gathered}
$$

The sign of the determinant Δ indicates the type of flow: on average, subsonic at $\Delta < 0$, and, on average, supersonic at $\Delta > 0$.

For constructing the solution of Eqs. (7), it is necessary to set the conditions at $x = x_0$, i.e., to find the asymptotics of the steady solution at $\eta \to 0$. It is assumed that the flow is supersonic and potential at the inlet to the channel. Using the second, fourth, and fifth equations of set (6), it is easy to determine the values of the functions v and q at $\eta \to 0$ (the corresponding values are designated by the subscript 0)

$$v_0 = \frac{u_0}{2+\kappa}, \quad q_0^2 = \frac{1+\kappa}{(2+\kappa)^2}u_0^2. \tag{8}$$

Here we assumed that the derivatives of the functions ρ, η, u, v, and q are finite at the point $x = x_0$; the coefficient c_f is assumed to be zero for simplicity.

In the region of the single-layer turbulent flow ($\eta = H_0 - z$), steady solutions (5) are found from the equations

$$
\begin{gathered}
\rho' = \frac{c_f v^3 - \sigma\kappa q^3 + (v^2 - 3q^2)vz'}{\Delta_1 v\eta},\\
\Delta_1 = v^2 - 3q^2 - c^2,\\
v' = \frac{v}{\eta}z' - \frac{v}{\rho}\rho', \quad q' = \frac{q}{\rho}\rho' - \frac{q}{\eta}z' - \frac{\sigma\kappa q^2}{2v\eta}.
\end{gathered} \tag{9}
$$

The sign of the value of Δ_1 determines the subsonic ($\Delta_1 < 0$) and supersonic ($\Delta_1 > 0$) flows. It is easy to understand that the density of the gas increases (decreases) in a channel of constant cross section in the absence of friction in the region of subsonic (supersonic) flow.

4. Further, we present three test calculations of flows on the basis of Eqs. (1). The tests include (i) comparison of the steady solution with the experimental data; (ii) implementation of the quasi-steady flow mode in which the pseudoshock position (the line $y = h$) coincides with the steady and unsteady equations in the calculations; (iii) the evolution of the pseudoshock under periodic variation of the outlet cross section of the channel.

4.1. We constructed the numerical solution of the problem on the interaction of a supersonic gas flow with a turbulent boundary layer within the framework of steady Eqs. (7). If the thickness of the potential supersonic core h vanishes at the point x_*, the solution in the region $x > x_*$ is constructed by the model of single-layer flow (9). On the input into the direct channel of the half-width $H_0 = 1$, the following parameters of the flow are set: $u_0 = 2$, $\rho_0 = 1$, and $c_0 = 1$ (dimensionless variables). The values of v and q at $x = x_0 = 2$ are set according to conditions (8), and $h_0 = H_0$, $\eta_0 = 0$. The calculations are carried out at $\gamma = 1.4$, $c_f = 0.04$, $\sigma = 0.3$, and $\kappa = 6$.

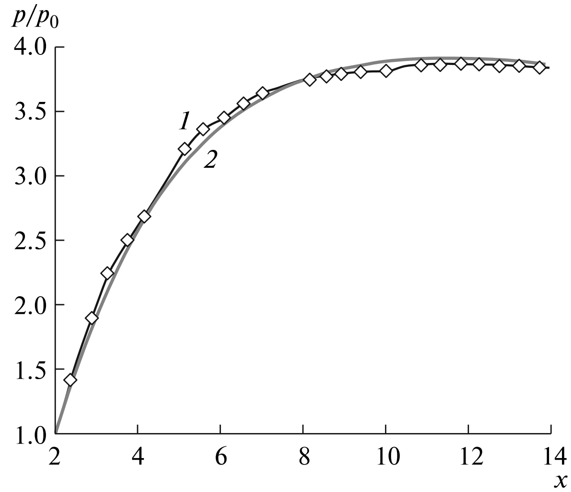

Fig. 1. Distribution of pressure along the channel length: *1*, experimental data [8]; *2*, solution of Eq. 9).

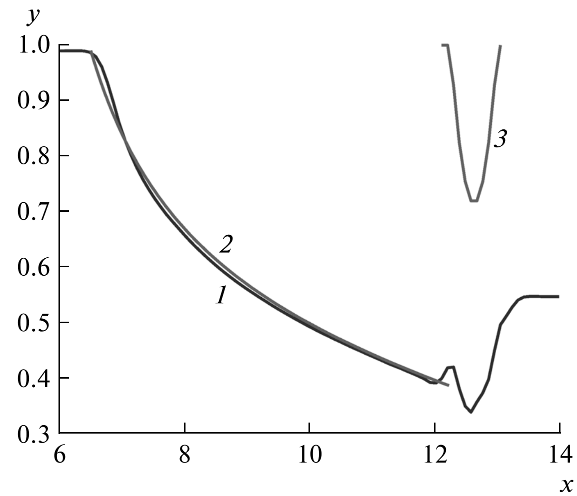

Fig. 2. Pseudoshock position $y = h$ (*1*, the solution of Eqs. (1); *2*, the steady solution; *3*, the shape of the obstacle).

The pressure distribution along the channel length is shown in Fig. 1, where curve *1* represents the experimental data [8, Fig. 2, p. 9] and curve *2* is the solution of Eqs. (7) and (9). At the specified parameters σ, κ, and c_f, the results of calculation agree well with the experiment. In the pseudoshock, we observed an increase in the pressure along the channel in the region of two-layer flow. As the boundary layers develop, the pressure growth is slowed down.

4.2. The quasi-steady pseudoshock is modeled on the basis of Eqs. (1). At $t = 0$, the supersonic flow having velocity $u = 2$ (the initial thickness of the turbulent layer $\eta = 0.02$, $v = u$, and $q = 0$) is set. In this example, $c_f = 0$, $\sigma = 0.15$, while γ and κ correspond to the previous test. On the inlet cross section of the channel $x = 0$, we used the initial data as the boundary conditions; on the right boundary $x = x_b = 14$, we used the conditions $\mathbf{u}_{N+1} = \mathbf{u}_N$ ($\mathbf{u}_i$ is the value of functions at the nodal point $x = x_i$). For carrying out the calculations, the Nessyahu–Tadmor scheme is applied [9] on a uniform grid of $N = 150$ nodes.

The formation of the pseudoshock is carried out due to the variation of the channel cross section. The function $z(t, x)$ responsible for it has the form $z = \max\{y_m - a_m(x - x_m)^2, 0\}$, where $a_m = 2$, $x_m = 0.9x_b$. The height of an obstacle varies jumpwise, $y_m = 0.4$ at $t < 2x_b$ and $y_m = 0.285$ at the subsequent moments of time. Such a choice of parameters provides for the formation of the quasi-steady pseudoshock, the position of which is shown in Fig. 2. Curve *1* corresponds to the function $y = h(t, x)$ obtained as a result of the solution of Eqs. (1) and deduced at $t = 150$. Curve *2* is obtained from the solution of Eqs. (7) with conditions (8) at the point $x_0 = 6.5$. The steady solution is constructed in the region before the local contraction of the channel, because at the transition through an obstacle, the value of Δ in Eqs. (7) vanishes (the flow passes from the supersonic mode into subsonic). For continuation of the steady solution, it is necessary to carry out the analysis in a vicinity of a special point. In the expanding turbulent layer, flow drag takes place and the gas velocity decreases substantially, which results in a transition, on average, to the subsonic flow in the vicinity of the obstacle. The steady and unsteady solutions in the region of the pseudoshock practically coincide before the obstacle.

4.3. Modeling of the evolution of an unsteady pseudoshock with periodic variation of the outlet cross section is carried out in a channel of length $L = 30$ and half-widths $H_0 = 1$ at $\gamma = 1.4$, $\sigma = 0.3$, $\kappa = 6$, and $c_f = 0$. The initial data correspond to the previous test: $\rho = 1$, $u = v = 2$, $\eta = 0.02$, and $q = 0$. At the outlet from the channel, throttling (periodic variation of the cross section) was carried out.

In Fig. 3 curve *1* represents the dependence $y = z_{\max}(t)/2$, where $z_{\max}$ is the highest value of the function $z(t, x)$ in the variable x. Within the framework of the model under consideration, the pseudoshock is formed at the extreme left point at which the thickness of the turbulent layer is larger than the set initial small value. We call this point the pseudo-shock front. Curve *2* in Fig. 3 represents the trajectory of displacement of the pseudo-shock front normalized to the channel length L. From the plot, it can be seen that the pseudo-shock periodically changes position. The results of calculation of the half-thickness $h(t, x)$ of the potential supersonic core are shown in Fig. 4. Curves *1* and *2* correspond to the largest deviations of the pseudoshock for one cycle of variation of the channel cross

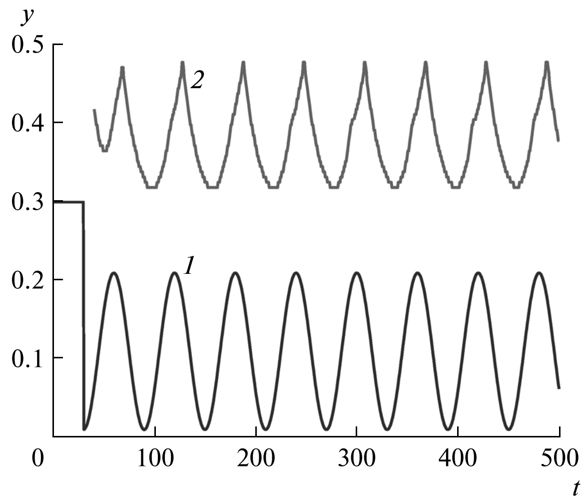

Fig. 3. Dependence $y = z_{max}(t)/2$ (curve *1*) and the trajectory of displacement of the pseudoshock front normalized to the channel length (curve *2*).

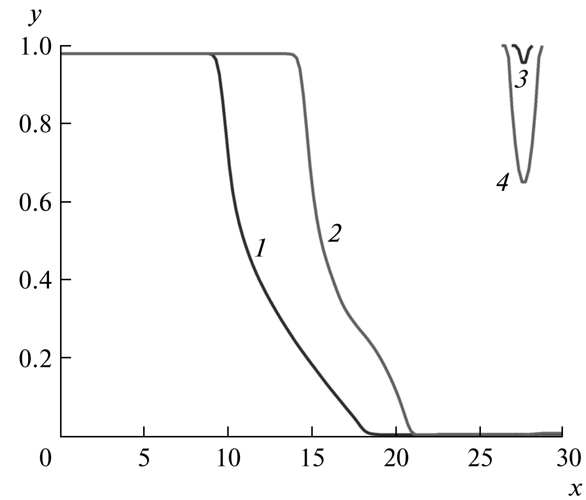

Fig. 4. Interface between the layers $y = h$ for the largest displacement of the pseudoshock (curves *1* and *2*; the obstacle shape $y = H_0 - z$ is shown by curves *3* and *4*).

section set by the equation $y = H_0 - z(t, x)$ (curves *3* and *4*).

Thus, within the framework of the two-layer scheme of flow, mathematical model (1) of unsteady pseudoshock in the flat long channel, which describes the continuous transition from the supersonic to subsonic flow, was constructed. This model does not yet pretend to be the final description of the processes of drag of supersonic flows. The issue is that, in addition to the forced vibrations of the pseudoshock in the experiments, the eigenvibrations or self-oscillations of the set of shocks as a whole were revealed, which is explained by the presence of inherent characteristic times or frequencies determined by unsteady processes of interaction. For the description of such effects, the proposed model should be complicated, particularly with respect to the description of the flow in the near-wall region. Such a model should contain an additional size (for example, the length of the separation zone) or an additional characteristic time.

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