In this work, we carried out the theoretical and experimental investigation of second-mode nonlinear internal waves on a thin interface between homogeneous layers of mixing fluids of different densities. We proposed a mathematical model describing the generation, interaction, and attenuation of the solitary internal waves arising during the intrusion of an intermediate-density fluid into the interlayer. The exact solution specifying the shape of solitary waves symmetric with respect to the unperturbed interface is constructed, and the passage to the limit for the finite-amplitude waves is justified for the case when the layer thickness tends to zero. It is shown that, taking into account the friction on interfaces in the mathematical model, it is possible to describe adequately the change in the phase and amplitude characteristics of two solitary waves moving towards each other before and after their interaction.

Intrusions propagating in a density-stratified fluid in the form of solitary waves are investigated in [1–8]. These waves are easily reproduced under laboratory and natural conditions and able to transfer mass due to an initial horizontal momentum along high-gradient interlayers in the stratified fluid. Here we investigated the plane-parallel flows caused by the intrusion of a finite volume of an intermediate-density fluid into a two-layer fluid at rest. The intrusion volume was selected so that the perturbation represented a solitary wave propagating along the interface of mixing fluids with shape preservation. For small differences in densities between the upper and lower layers and an identical initial thickness of layers, the perturbation is represented by a second-mode internal solitary wave symmetric with respect to the unperturbed interface. We determined the initial interlayer thickness $\delta$ and the wave amplitude $A$ in the experiments on high-gradient layers visualized in photographs.

The experiments were carried out in a transparent laboratory tray of 160 cm in length, 20 cm in width, and 45 cm in depth [9]. The face parts of the tray of 20 cm in length were separated by hermetic partitions, which could be elevated and lowered for the formation of solitary waves. As working fluids, we used fresh water and a weak solution of sugar in water. The installation was filled with layers of these fluids of equal thickness $H = 6$ cm. When removing the partition, the intermediate-density fluid propagated from the section along the layer interface in the form of the second-mode solitary wave. If necessary, two waves moving towards each other were formed. The flow visualization is portrayed in Fig. 1, where we show a photograph of the generated solitary wave against the background of the luminous screen with a grid of inclined lines imposed on it. The optical inhomogeneities related to the perturbation propagation are distinctly seen. In high-density-gradient regions, we

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**Fig. 1.** Solitary wave on the interface of fluids.
observed the characteristic distortion of lines, whereas the optical transparency of the fluid changes in the mixing areas [10]. In addition, the intermediate-density fluid, which propagates along the interface in the form of an intrusion, is slightly tinted with an ink solution for visualizing the mass-transfer processes. The external wave boundaries are determined by the break and change in thickness of the inclined lines. The bold line represents the exact solution (soliton) constructed from the set wave amplitude. From Fig. 1, it is seen that the constructed solution adequately represents the external boundaries of the density-field perturbations during the solitary-wave propagation.

When constructing the mathematical model of propagation of intrusions in a finite-depth channel, the approach described in [11] is used. In the long-wave approximation, we consider a three-layer scheme of stratified-fluid flow in which two layers of the homogeneous fluid of the densities \( \rho^+ \) and \( \rho^- \) are separated by an interlayer with the average density \( \rho = (1/2)(\rho^+ + \rho^-) \). In homogeneous layers, the nonhydrostatic character of the pressure distribution is taken into account within the second approximation of the shallow-water equations, whereas the pressure in the interlayer is considered as distributed according to the hydrostatic law. The last hypothesis is applicable to large-amplitude waves (\( A/\delta \gg 1 \)) or to the so-called “waves with a captured nucleus,” in which the major fluid mass in the interlayer is transferred with a rate close to the wave velocity.

In the Boussinesq approximation (\( 0 < (\rho^+ + \rho^-)/\rho^+ \ll 1 \)), the second-mode internal waves are symmetric with respect to the channel midline \( y = H \) for the symmetric initial stratification. Therefore, it suffices to consider only the flow in the lower half of the channel for describing the symmetric-wave propagation in the interlayer. Let the wave profile (the depth of the perturbed homogeneous layer) be set by the line \( y = h(t, x) \), \( u = u(t, x) \) is the average velocity of the lower homogeneous layer, \( w = w(t, x) \) is the average velocity of the fluid intrusion, and \( g \) is the gravity acceleration. In the dimensionless variables (\( H = 1, \rho^+ = 1, (\rho^- - \rho^+)g = 1 \)), the set of equations of motion takes the form of

\[
\eta_t + (\eta w)_x = 0,
\]

\[
w_t + \left( \frac{w^2}{2} - hu - \eta w^2 \right)_x = -\frac{c(w-u)(w-u)}{\eta}, \quad c = \text{const},
\]

\[ hu + \eta w = 0, \quad h + \eta = 1. \tag{1} \]

Set (1) is the variant of the Green–Nagdi equations and follows from the equations for two-layer shallow water [12, Eqs. (3.19)–(3.23)] with taking into account the hydrostatic character of the pressure in the upper layer and the friction on the layer interface.

We consider waves (1) running on a fluid originally at rest for \( c = 0 \), i.e., the solutions dependent only on the variable \( \xi = x - Dr \) and \( D = \text{const} \). In the system of coordinates moving with the wave, Eqs. (1) become

\[
hu = h_0 D, \]

\[
\eta w = \eta_0 D, \]

\[
\frac{1}{2} w^2 + p = \frac{1}{2} D^2 + p_0, \tag{2} \]

\[ hu^2 + \eta w^2 + \frac{1}{2} h^2 + p + \frac{1}{3} \delta u(uh)' = h_0 D^2 + \frac{1}{2} h_0^2 + p_0. \]

With taking into account the asymptotic behavior

\[ h \to h_0, \quad h' \to 0, \quad h'' \to 0 \quad \text{при} \quad |x| \to \infty \]

for the set lower layer depth \( h_0 \) and the Froude number \( Fr = D \), Eqs. (2) can be integrated:

\[
(h')^2 = G_0(h) = \frac{3h^2}{h_0^2 Fr} \left( \left( \frac{h_0^2 + Fr^2}{h_0^2} \right) \frac{1}{h} - \frac{1}{h_0} \right) + h_0
\]

\[
- h + \frac{Fr^2 h_0}{h^2} - Fr^2 + \frac{(1 - h_0^2) Fr^2}{(1 - h)} - \frac{(1 - h_0) Fr^2}{h_0}. \tag{3} \]

At \( h_0 \to 1 \), the function \( G_0(h) \) is represented as

\[
G_1(h) = \frac{3h^2}{Fr^2} \left( \left( 1 + Fr^2 \right) \frac{1}{h} - 1 + h + Fr^2 \left( \frac{1}{h^2} - 1 \right) \right), \tag{4} \]

from which the solitary wave propagating on the zero-thickness interlayer (\( A/\delta \to 0 \)) can be found. The behavior of the functions \( G_0(h) \) for \( h_0 < 1 \) and \( G_1(h) \) is shown in Fig. 2. Let \( h_m \) be the largest root of the equation \( G_0(h) = 0 \) such that \( h_m < h_0 \). An interesting feature of model (1) is the independence of \( h_m \) on \( h_0 \). This means that the solitary-wave amplitude for the fixed Froude number \( Fr \) is independent of the initial thickness of the interlayer and can be found explicitly from the equation \( G_1(h) = 0 \):

\[
h_m = \frac{1 + \sqrt{1 - 4 Fr^2}}{2}. \tag{5} \]

We recall that model (1) is reduced under the condition of

\[
\frac{A \eta_0}{1 - h_0} \gg 1; \text{ therefore, it can be used in describing the finite-amplitude waves only for a relatively thin initial layer. The wave profile is found in quadratures both for the finite-thickness layer (}h_0 < 1\text{) and for the zero-thickness layer (}h_0 = 1\text{) i.e.,}
\]

\[
x = \pm \int_{h_0}^{h} \frac{ds}{\sqrt{G_1(s)}}, \quad i = 0, 1. \tag{6} \]
We note that the restriction $F_r < F_{r*} = 0.5$ follows from Eq. (5), and Eqs. (3) and (4) lead to the following properties of the functions $G_i(h)$ for $i = 0, 1$:

$$
G_0(h_0) = 0, \quad G_0'(h_0) = 0 \quad \text{for} \quad h_0 < 1,
$$

$$
G_1(1) = 0, \quad G_1'(1) > 0.
$$

Thus, solution (3) constructed from Eq. (6) is a soliton for $h_0 < 1$ and a compacton, i.e., the solution with the compact carrier with respect to the function $h(\xi) - h_0$, for $h_0 = 1$. Solutions (6) can be used due to their simplicity both for describing the symmetric solitary waves and for representing the intrusion front in real stratified fluids. In Fig. 1, we show steady-state solutions (3) with $h_0 < 1$ (curve 2) and $h_0 = 1$ (curve 1) together with the photograph of the internal solitary wave realized in the experiment. We note that the constructed solutions are determined (for set $h_0$) by the Froude number $F_r$ occurring in a relatively small vicinity of the limiting value $F_{r*} = 0.5$ ($F_r = 0.48$ in Fig. 1) for finite-amplitude waves. In this case, the solution with nonzero initial interlayer thickness (curve 2) precisely reproduces the external interface of the density-field perturbations during the wave passage despite the relatively small wave amplitude $(A/\eta_0 = 1 - h_0 \approx 2.6)$.

In a real system with dissipation, which can be a stratified fluid, solitary waves propagating with a constant velocity cannot be realized. During the propagation, the solitary wave loses mass and its velocity decreases. Nevertheless, the motion of an internal wave is quasi-steady: at each moment of time, its shape coincides with that of the stationary wave of the same amplitude. The discrepancies are observed only at the small portion adjoining to the area of constant flow behind the wave (Fig. 1).

The problem about the interaction of the solitary waves moving towards each other can be solved numerically as the Cauchy problem for set (1) with the initial data in the form of solitary waves (3) propagating with the velocities $D_1 > 0$ and $D_2 < 0$ and carried on a sufficient distance so that the unperturbed stratified fluid was between them. In Fig. 3a we show the main stages of interaction between the solitary waves realized experimentally. In Fig. 3b, we show the consecutive positions of waves before and after the interaction calculated through the regular intervals of time (2 s). The bold points indicate the wave-crest location constructed numerically, and the empty circles are those found experimentally. It is seen that the model quali-
tatively and quantitatively reflects the change in the phase and amplitude characteristics of solitary waves before and after their interaction. The numerical value of the friction coefficient $c = 0.01$ gives satisfactory results for various initial interlayer thicknesses and amplitudes of the solitary waves realized experimentally.

Thus, model (1) adequately represents the structure of solitary symmetric large-amplitude waves propagating on the interface of mixing fluids and is also suitable for description of the unsteady interaction of such waves. Constructed analytical solution (3) can be used for simulating the near-surface internal large-amplitude waves.

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