Parabolic equations with double variable nonlinearities

S. Antontseva\textsuperscript{a,1}, S. Shmarev\textsuperscript{b,*,2}

\textsuperscript{a} CMAF, University of Lisbon, Portugal
\textsuperscript{b} Department of Mathematics, University of Oviedo, Spain

Received 15 November 2009; received in revised form 26 May 2010; accepted 7 December 2010
Available online 21 December 2010

Abstract

The paper is devoted to the study of the homogeneous Dirichlet problem for the doubly nonlinear parabolic equation with nonstandard growth conditions:

\[ u_t = \text{div} \left( a(x, t, u)|u|^{\alpha(x,t)}|\nabla u|^{p(x,t)-2}\nabla u \right) + f(x, t) \]

with given variable exponents \( \alpha(x, t) \) and \( p(x, t) \). We establish conditions on the data which guarantee the existence of bounded weak solutions in suitable Sobolev–Orlicz spaces.

© 2011 IMACS. Published by Elsevier B.V. All rights reserved.

MSC: 35K57; 35K65; 35B40

Keywords: Parabolic equation; Double nonlinearity; Variable nonlinearity; Nonstandard growth conditions

1. Introduction

We study the Dirichlet problem for the doubly nonlinear parabolic equation

\[
\begin{aligned}
  &u_t = \text{div} \left( a(z, u)|u|^{\gamma(z)}|\nabla u|^{p(z)-2}\nabla u \right) + f(z) & z = (x, t) \in Q = \Omega \times (0, T], \\
  &u(x, 0) = u_0(x) & \text{in} \Omega, \\
  &u = 0 & \text{on} \Gamma = \partial\Omega \times [0, T].
\end{aligned}
\] (1.1)

Eq. (1.1) is formally parabolic, but it may degenerate or become singular at the points where \( u = 0 \) or \( |\nabla u| = 0 \).

Introducing the functions

\[
\gamma(z) = \frac{\alpha(z)}{p(z) - 1}, \quad v(z) = \int_0^u |s|^{\gamma(z)} ds = \frac{u|u|^{\gamma(z)}}{\gamma(z) + 1}, \quad u(z) = \Phi_0(z, v) = \frac{|v|^{\gamma(z)}}{(1 + \gamma)^{\frac{1}{p(z)}}}.
\] (1.2)

* Corresponding author.

E-mail address: shmarev@orion.ciencias.uniovi.es (S. Shmarev).

\textsuperscript{1} The author was partially supported by FCT, Financiamento Base 2008-ISFL-1-209, Portugal, and by the research project MTM2008-06208 of the Ministerio de Ciencia e Innovacion, Spain.

\textsuperscript{2} The author acknowledges the support of the research project MTM2007-65088, Spain.

0378-4754/$36.00 \copyright 2011 IMACS. Published by Elsevier B.V. All rights reserved.
we write problem (1.1) in the following form:

\[
\begin{aligned}
\partial_t \Phi_0(z, v) &= \text{div} \left( b(z, v) \nabla v + B(z, v) \right) + f \quad \text{in } Q, \\
v(x, 0) &= v_0(x) = \frac{u_0 |u_0|^\gamma(x, 0)}{1 + \gamma(x, 0)} \quad \text{in } \Omega, \\
v = 0 &\quad \text{on } \Gamma
\end{aligned}
\]  

(1.3)

with

\[
\begin{aligned}
b(z, v) &= a(z, \Phi_0(z, v)), \\
B(z, v) &= -\nabla \gamma(z) \int_0^{\mu(z)} |s|^\gamma(z) \ln |s| \, ds.
\end{aligned}
\]

Problem (1.3) will be the subject of the further study. Equations of the types (1.1) and (1.3) with constant exponents $\alpha$ and $p$ arise in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology – see [2,5,14,15,21,34,6,27,28] and the further references therein. The questions of existence and uniqueness of solutions to equations like (1.1) and (1.3) with constant exponents of nonlinearity $\alpha$ and $p$ were studied by many authors – see [14,17,23,31,33] for equations of the type (1.1) and [14,15,20] for the equations of the type (1.3) with constant exponent $p$. In the present work we prove the existence theorem for the Dirichlet problem (1.3) in which the exponents $\alpha$ and $p$ are allowed to be variable.

Existence, uniqueness, and qualitative properties of solutions for parabolic equations with variable nonlinearity corresponding to the special cases $\alpha(x, t) = 0$ or $p(x, t) = 2$ were studied in [1,3,4,8–10,12], see also [7] for a review of the results concerning elliptic equations with variable nonlinearity.

The notion of solution to problem (1.3) is introduced in Definition 3.1 below; this is the so-called energy solution. Another notion of solution was adopted in [1] (see Remark 4.1 below). Relations between different definitions of solution to the evolution $p(x, t)$-Laplace equation in dependence of the regularity of the variable exponent $p(x, t)$ are discussed in [1]. Existence and uniqueness of the renormalized and entropy solutions to elliptic equations of the $p(x)$-Laplace type and the evolution $p(x)$-Laplace equation are proved in [13,35,36]. The question of existence of such solutions is still open for evolution $p$-Laplace equation with the exponent $p$ depending on $t$, as well as for the doubly nonlinear elliptic and parabolic equations with variable exponents of nonlinearity.

The paper is organized as follows. In Section 2 we collect some known facts from the theory of Orlicz–Sobolev spaces and prove several auxiliary assertions. In particular, we derive the formula of integration by parts, which is required to deal with the nonlinear term $\partial_t \Phi_0(z, v)$. The precise assumptions on the data and the main result are given in Section 3. A solution to problem (1.3) is obtained as the limit of solutions of the regularized problems for the evolution $p(z)$-Laplace equations already studied in [1,10]. In Section 4 we define the sequence of regularized problems, whose solutions belong to suitable Orlicz–Sobolev spaces, and derive a series of a priori estimates. In Section 5 we prove, relying on the a priori estimates of Section 4, that the constructed sequence converges to a weak solution of problem (1.3).

The authors would like to express their gratitude to Professor M. Chipot for stimulating discussions of this work.

2. The function spaces

2.1. Spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}_0(\Omega)$

The definitions of the function spaces used throughout the paper and a brief description of their properties follow [16,18,22,25]. The further references can be found in the review papers [19,29]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial \Omega$ be Lipschitz-continuous, and let $p(x)$ be log-continuous in $\Omega$: $\forall x, y \in \Omega$ such that $|x - y| < \frac{1}{2}$

\[
|p(x) - p(y)| \leq \omega(|x - y|) \quad \text{with} \quad \lim_{\tau \to 0^+} \left( \omega(\tau) \ln \frac{1}{\tau} \right) = C < \infty.
\]  

(2.1)

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on $\Omega$ such that

\[
A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} \, dx < \infty.
\]
The set $L^{p(\cdot)}(\Omega)$ equipped with the norm
\[
\|f\|_{p(\cdot), \Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : A_{p(\cdot)}(f/\lambda) \leq 1 \right\}
\]
becomes a Banach space. The Banach space $W_{0}^{1,p(\cdot)}(\Omega)$ with $p(x) \in [p^{-}, p^{+}] \subset (1, \infty)$ is defined by
\[
\begin{align*}
W_{0}^{1,p(\cdot)}(\Omega) &= \left\{ f \in L^{p(\cdot)}(\Omega) : |\nabla f|^{p(\cdot)} \in L^{1}(\Omega), \ u = 0 \quad \text{on} \ \partial \Omega \right\}, \\
\|u\|_{W_{0}^{1,p(\cdot)}(\Omega)} &= \sum_{i} \|D_{i} u\|_{p(\cdot), \Omega} + \|u\|_{p(\cdot), \Omega}.
\end{align*}
\tag{2.2}
\]

Throughout the paper we use the following properties of the functions from the spaces $W_{0}^{1,p(\cdot)}(\Omega)$:

- if condition (2.1) is fulfilled, then $C_{0}^{\infty}(\Omega)$ is dense in $W_{0}^{1,p(\cdot)}(\Omega)$, and the space $W_{0}^{1,p(\cdot)}(\Omega)$ can be defined as the closure of $C_{0}^{\infty}(\Omega)$ with respect to the norm (2.2) – see [30,37–39];
- if $p(x) \in C^{0}(\bar{\Omega})$, then the space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive;
- if $1 < q(x) \leq \sup_{\Omega} q(x) < \inf \limits_{\Omega} p_{+}(x)$ with
\[
p_{+}(x) = \begin{cases} 
p(x)n & \text{if } p(x) < n, \\
\infty & \text{if } p(x) > n,
\end{cases}
\tag{2.3}
\]
then the embedding $W_{0}^{1,p(\cdot)}(\Omega) \rightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact;
- it follows directly from the definition that
\[
\min \left( \|f\|_{p^{-}, p(\cdot)}^{p^{-}}, \|f\|_{p^{+}, p(\cdot)}^{p^{+}} \right) \leq A_{p(\cdot)}(f) \leq \max \left( \|f\|_{p^{-}, p(\cdot)}^{p^{-}}, \|f\|_{p^{+}, p(\cdot)}^{p^{+}} \right);
\tag{2.4}
\]
- for all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p^{'}(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$, $p^{'} = \frac{p}{p-1}$ Hölder’s inequality holds,
\[
\int_{\Omega} \frac{fg}{dx} \leq \left( \frac{1}{p^{-}} + \frac{1}{p^{'}-} \right) \|f\|_{p(\cdot)} \|g\|_{p^{'}(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{p^{'}(\cdot)};
\tag{2.5}
\]

2.2. Parabolic spaces $L^{p(\cdot)}(Q)$ and $W(Q)$

Let $p(z, z = (x, t) \in Q$, satisfy condition (2.1) in the cylinder $Q$. For every fixed $t \in [0, T]$ we introduce the Banach space
\[
V_{t}(\Omega) = \{ u(x) : u(x) \in L^{2}(\Omega) \cap W_{0}^{1,1}(\Omega), \ |\nabla u(x)|^{p(x,t)} \in L^{1}(\Omega) \},
\|
\|u\|_{V_{t}(\Omega)} = \|u\|_{2, \Omega} + \|\nabla u\|_{p(\cdot,t), \Omega},
\]
and denote by $V_{t}^{'}(\Omega)$ its dual. By $W(Q)$ we denote the Banach space
\[
\begin{align*}
W(Q) &= \{ u : [0, T] \mapsto \forall t \in V_{t}(\Omega) | u \in L^{2}(Q), \ |\nabla u|^{p(x,t)} \in L^{1}(Q), \ u = 0 \text{ on } \Gamma \}, \\
\|u\|_{W(Q)} &= \|\nabla u\|_{p(\cdot,t), Q} + \|u\|_{2, Q}.
\end{align*}
\tag{2.6}
\]
$W^{'}(Q)$ is the dual of $W(Q)$ (the space of linear functionals over $W(Q)$:
\[
w \in W^{'}(Q) \iff \begin{cases} w = w_{0} + \sum_{i=1}^{n} D_{i} w_{i}, w_{0} \in L^{2}(Q), w_{i} \in L^{p^{'}(\cdot)}(Q), \\
\forall \phi \in W(Q)(\langle w, \phi \rangle) = \int_{Q} \left( w_{0}\phi + \sum_{i} w_{i} D_{i} \phi \right) dz.
\end{cases}
\]
The norm in $W'(Q)$ is defined by
\[ ||v||_{W'(Q)} = \sup \{ \langle \langle v, \phi \rangle \rangle | \phi \in W(Q), \|\phi\|_{W(Q)} \leq 1 \}. \]

Set
\[ V_+ = \left\{ u(x) | u \in L^2(\Omega) \cap W^{1,1}_0(\Omega), |\nabla u| \in L^{p'}(\Omega) \right\}. \]

Since $V_+$ is separable, it is a span of a countable set of linearly independent functions $\{ \psi_k(x) \} \subset V_+$. Without loss of generality, we may assume that this system forms an orthonormal basis of $L^2(\Omega)$.

**Lemma 2.1 ([10]).** Let conditions (2.1) hold. Then the set $\{ \psi_k \}$ is dense in $V_t(\Omega)$ for every $t \in [0, T]$.

**Lemma 2.2 ([10]).** For every $u \in W(Q)$ there is a sequence $\{ d_k(t) \}, d_k(t) \in C^1[0, T]$, such that
\[ \| u - \sum_{k=1}^m d_k(t) \psi_k(x) \|_{W(Q)} \to 0 \quad \text{as} \quad m \to \infty. \]

### 2.3. Formulas of integration by parts

Let $\rho$ be the Friedrichs mollifying kernel
\[ \rho(s) = \begin{cases} \kappa \exp \left( -\frac{1}{1 - |s|^2} \right) & \text{if } |s| < 1, \\ 0 & \text{if } |s| > 1, \end{cases} \quad \kappa = \text{const} : \int_{\mathbb{R}^{n+1}} \rho(z)dz = 1. \]

Given a function $v \in L^1(Q_T)$, we extend it to the whole $\mathbb{R}^{n+1}$ by a function with compact support (keeping the same notation for the continued function) and then define
\[ v_h(z) = \int_{\mathbb{R}^{n+1}} v(s) \rho_h(z - s)ds \quad \text{with} \quad \rho_h(s) = \frac{1}{h^{n+1}} \rho \left( \frac{s}{h} \right), \quad h > 0. \]

**Lemma 2.3.** If $u \in W(Q_T)$ with the exponent $p(z)$ satisfying (2.1) in $Q$, then
\[ \| u_h \|_{W(Q)} \leq C \left( 1 + \| u \|_{W(Q)} \right) \quad \text{and} \quad \| u_h - u \|_{W(Q)} \to 0 \quad \text{as} \quad h \to 0. \]

**Lemma 2.3** is an immediate byproduct of [39,Theorem 2.1].

**Lemma 2.4 ([10]).** Let in the conditions of Proposition $u_t \in W'(Q)$. Then $(u_h)_t \in W'(Q)$, and for every $\psi \in W(Q)$
\[ \langle \langle (u_h)_t, \psi \rangle \rangle \to \langle \langle u_t, \psi \rangle \rangle \quad \text{as} \quad h \to 0. \]

**Lemma 2.5 (Integration by parts).** Let $v, w \in W(Q)$ and $v_t, w_t \in W'(Q)$ with the exponent $p(z)$ satisfying (2.1) in $Q$. Then
\[ \forall a.e. \quad t_1, t_2 \in (0, T) \quad \int_{t_1}^{t_2} \int_{\Omega} v w_tdz + \int_{t_1}^{t_2} \int_{\Omega} v_t wdz = \int_{\Omega} v wdx \big|_{t=t_2}^{t-t_1}. \]
Proof. Let \( t_1 < t_2 \). Take
\[
\chi_k(t) = \begin{cases} 
0 & \text{for } t \leq t_1, \\
k(t - t_1) & \text{for } t_1 \leq t \leq t_1 + \frac{1}{k}, \\
1 & \text{for } t_1 + \frac{1}{k} \leq t \leq t_2 - \frac{1}{k}, \\
k(t_2 - t) & \text{for } t_2 - \frac{1}{k} \leq t \leq t_2, \\
0 & \text{for } t \geq t_2.
\end{cases}
\]
(2.7)
For every \( k \in \mathbb{N} \) and \( h > 0 \)
\[
0 = \int_Q (v_h h \chi_k)_t dz = \int_Q (v_h w_h)_t \chi_k dz - k \int_0^{\theta} \int_\Omega v_h w_h \chi_{k, t}^\theta dz.
\]
The last two integrals on the right-hand side exist because \( v_h, w_h \in L^2(Q_T) \). Letting \( h \to 0 \), we obtain the equality
\[
\lim_{h \to 0} \int_Q (v_h (w_h)_t + (v_h)_t w_h) \chi_k(t) dz = k \int_{t_2}^{t_2} \int_\Omega v w dz - k \int_{t_1}^{t_1 + \frac{1}{k}} \int_\Omega v w dz.
\]
According to Lemmas 2.3 and 2.4 \( v_h \to v \) in \( W(Q) \), \( (w_h)_t = (w_t)_h \to w_t \) weakly in \( W'(Q_T) \) as \( h \to 0 \), and \( \|v\|_W, \|\|w_h\|_W \) are uniformly bounded. It follows that
\[
\lim_{h \to 0} \int_Q v_h (w_h)_t \chi_k(t) dz = \lim_{h \to 0} \int_Q (v_h - v)(w_h)_t \chi_k(t) dz + \lim_{h \to 0} \int_Q v(w_h)_t - w_t \chi_k(t) dz + \int_Q v w_t \chi_k(t) dz = \int_Q v w_t \chi_k(t) dz.
\]
In the same way we check that
\[
\lim_{h \to 0} \int_Q v_h w_h \chi_k(t) dz = \int_Q v w_t \chi_k(t) dz.
\]
By the Lebesgue differentiation theorem
\[
\forall a.e. \theta > 0 \quad \lim_{k \to 0} k \int_{\theta - \frac{1}{k}}^{\theta} \left( \int_\Omega v w dz \right) dt = \int_\Omega v w dz,
\]
whence for almost every \( t_1, t_2 \in [0, T] \)
\[
\int_{t_1}^{t_2} \int_\Omega (v w_t + v_t w) dz = \lim_{k \to \infty} \int_{\theta - \frac{1}{k}}^{\theta} \left( \int_\Omega (v w_t + v_t w) \chi_k(t) dz \right) dt = \lim_{k \to \infty} \int_{\theta - \frac{1}{k}}^{\theta} \frac{1}{k} \int_\Omega v w dx |_{\theta = t_1}^\theta = \int_\Omega v w dx |_{\theta = t_1}^{t_2}.
\]
Corollary 2.1. Let \( u \in W(Q) \) and \( u_t \in W'(Q) \) with the exponent \( p(z) \) satisfying (2.1). Then
\[
\forall a.e. \quad t_1, t_2 \in (0, T] \quad \int_{t_1}^{t_2} \int_\Omega u u_t dz = \frac{1}{2} \|u\|_{2, Q|_{t=t_2}}.
\]
We will need two elementary inequalities.

Proposition 2.1 ([14]). For every \( p \geq 2, k \geq |b| \geq 0 \)
\[
|a|^{p-2} a - |b|^{p-2} b \leq C(p) |a - b| (|a| + |b|)^{p-1}.
\]
This proposition is an immediate byproduct of the easily verified relation

\[ 1 - t^{p-1} \leq C(p)(1 - t)(1 + t)^{p-1} \quad \forall p \geq 2, \quad t \in [0, 1]. \]

**Proposition 2.2 ([14]).** For \( 2 - p < \beta < 1 \) and \(|a| \geq |b| \geq 0\)

\[ ||a|^{p-2}a - |b|^{p-2}b| \leq C(p)\max(|a|, |b|)^{1-\beta} + |a|^{p-2}\beta. \]

The assertion follows from the inequality

\[ 1 - t^{p-1} \leq C(p)(1 - t)^{1-\beta}(1 + t)^{p-2}\beta, \quad t \in [0, 1] \]

with the same \( p \) and \( \beta \).

**Lemma 2.6.** Let \( u \in W(Q) \cap L^\infty(Q) \), \( u_t \in W'(Q) \), and let the exponent \( p(z) \) satisfy (2.1). Introduce the function

\[ v = \int_0^u \left( \varepsilon + |\sigma| \right) \gamma(z) \sigma \, ds, \quad \varepsilon > 0, \]

with the exponent \( \gamma(z) \geq \gamma^- \geq 1 \) such that \( \gamma \gamma \in L^2(Q) \) and \( |\nabla\gamma(z)| \gamma(z) \in L^1(Q) \). For a.e. \( t_1, t_2 \in [0, T] \)

\[
\int_{t_1}^{t_2} \int_\Omega u_t v \, dz = \int_\Omega \frac{uv}{\gamma + 2} \int_{t_1}^{t_2} \gamma \, dz + \int_{t_1}^{t_2} \int_\Omega \frac{v}{\gamma + 2} \, dx \, dz + \varepsilon \int_{t_1}^{t_2} \int_\Omega \frac{v}{\gamma + 2} \, dx \, dz \equiv \mu_\varepsilon(u, v). \tag{2.8}
\]

**Proof.** Let \( u_h \in C^\infty(Q) \) be the mollification of \( u \in W(Q) \) and

\[ v_h = \int_0^{u_h} \left( \varepsilon + |\sigma| \right) \gamma(z) \sigma \, ds = \frac{\varepsilon}{\gamma + 1} \left( (\varepsilon + |u_h|)^{\gamma + 1} - |u_h|^{\gamma + 1} \right). \]

Since \( u \) and \( u_h \) are bounded by a constant \( K_0 \), and \( \gamma(z) \geq \gamma^- \geq 1 \), it follows from Lemmas 2.3 and 2.4 that

\[ |v_h - v| \leq C \max \left\{ |u_h - u|, |u_h - u|^{|1+\min(0, \gamma^- - 1)} \right\}, \quad C \equiv C(\varepsilon, p^\pm, \alpha^\pm, K_0). \]

The inclusion \( u \in L^\infty(Q) \) entails the convergence \( ||v_h - v||_{L^\infty(Q)} \to 0 \) as \( h \to 0 \) for every \( s > 1 \). Explicitly calculating the primitive, in the same way we check that for every \( s > 1 \)

\[ \left\| \int_\Omega \left( \varepsilon + |\sigma| \right) \gamma(z) \sigma \, ds \right\|_{L^\infty(Q)} \to 0 \quad \text{as} \quad h \to 0. \]

Let \( \psi_k(z) = \frac{\psi(k)}{\gamma_+} \) with the function \( \chi_k \) introduced in (2.7). Following the proof of Lemma 2.5, we find:

\[
k \int_{t_1}^{t_2} dt \int_\Omega \frac{u_h v_h}{\gamma + 2} \, dx \bigl|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_\Omega \chi_k(t)(u_h)v_h \, dz - \int_{t_1}^{t_2} \int_\Omega \frac{u_h v_h}{\gamma + 2} \gamma \chi_k(t) \, dz \]

\[ -\varepsilon \int_{t_1}^{t_2} \int_\Omega \chi_k(t) \frac{\gamma}{\gamma + 2} \int_0^{u_h} (\varepsilon + |\sigma|) \gamma \sigma \, ds \, dz \]

\[ +\varepsilon \int_{t_1}^{t_2} \int_\Omega \chi_k(t) v_h \frac{\gamma}{\gamma + 2} \, dz + \varepsilon k \int_{t_1}^{t_2} dt \int_\Omega \frac{v_h}{\gamma + 2} \, dx \bigl|_{t_1}^{t_2}. \tag{2.9}
\]

Since \( u \in W(Q) \cap L^\infty(Q) \) and \( \gamma^- > -1, v \in W(Q) \) for every \( \varepsilon > 0 \). Indeed: since \( ||u||_{L^\infty(Q)} \leq M \), we have the estimates

\[ |v| \leq M_1(\varepsilon^\pm, M), \quad \int_0^{t_1} \left( \varepsilon + |\sigma| \right) \gamma \sigma ds \leq M_2(\varepsilon^\pm, M), \]
which provide the inclusion
\[ |\nabla v| \leq (e + |u|)^{p(z)} |\nabla u| + |\nabla y| \int_0^{[u]} (e + |s|)^{y(z)} \ln(e + |s|)ds \in L^{p(z)}(Q). \]

By Lemma 3
\[ \|v_h\|_{W(Q)} \leq C(1 + \|v\|_{W(Q)}) \text{ and } \|v_h - v\|_{W(Q)} \to 0 \text{ as } h \to 0. \]

We may now pass to the limit as \( h \to 0 \) in every term of (2.9), following the proof of Lemma 2.5:
\[ k \int_{\theta-1}^{0} \frac{1}{k} dt \int_{\Omega} \frac{uv}{\gamma + 2} dx^{[t=1]} = \int_{t_1}^{t_2} \int_{\Omega} \chi(t) u_1 v_1 dz - \int_{t_1}^{t_2} \int_{\Omega} \frac{uv}{\gamma + 2} \gamma_1 \chi(t) dz \]
\[ -\varepsilon \int_{t_1}^{t_2} \int_{\Omega} \chi(t) \frac{\gamma_1}{\gamma + 2} \int_{0}^{u} (e + |s|)^{y} \ln(e + |s|)ds dz + \varepsilon \int_{t_1}^{t_2} \int_{\Omega} \chi(t) v_1 \frac{\gamma_1}{(\gamma + 2)^2} dz + e \int_{\theta-1}^{0} dt \int_{\Omega} \frac{v}{\gamma + 2} dx^{[t=1]}. \]

Letting \( k \to \infty \) and applying the Lebesgue differentiation theorem, we arrive at (2.8).

**Remark 2.1.** Let \( \varepsilon = 0, u \in W(Q), u_1 \in W^1(Q), \) and let \( v = \frac{u_1}{\gamma + 1} \in W(Q). \) Under the foregoing conditions on the exponents \( p(z) \) and \( y(z) \) the following formula of integration by parts holds:
\[ \forall a.e. t_1, t_2 \in [0, T], \int_{t_1}^{t_2} \int_{\Omega} u_1 v_1 dz = \int_{t_1}^{t_2} \int_{\Omega} \frac{uv}{\gamma + 2} dx^{[t=1]} + \int_{t_1}^{t_2} \int_{\Omega} \frac{uv}{\gamma + 2} \gamma_1 dz = \mu(u, v). \]

**3. Assumptions and results**

The existence result is established for the problem
\[ \begin{align*}
\partial_t \Phi_0(z, v) &= dv \left( b(z, v) |\nabla v + B(z, v)|^{p(z)-2} (\nabla v + B(z, v)) \right) + f \quad \text{in } Q, \\
v(x, 0) &\text{ in } \Omega, \quad v = 0 \quad \text{on } \Gamma, 
\end{align*} \tag{3.1} \]

which is formally equivalent to problem (1.1). Throughout the paper we assume that the coefficient \( a(z, r) \) and the exponents on nonlinearity \( p(z), \alpha(z) \) satisfy the following conditions:

- \( a(z, r) \) is a Carathéodory function, there exists positive constants \( a^\pm \) such that
  \[ \forall z \in Q, r \in \mathbb{R}, \quad a^- \leq a(z, r) \leq a^+ < \infty, \] \hfill (3.2)

- \( \alpha(z) \), \( p(z) \) are measurable and bounded in \( Q \), there exist constants \( a^\pm, p^\pm \) such that
  \[ -1 < a^- \leq \alpha(z) \leq a^+ < \infty, \quad 1 < p^- \leq p(z) \leq p^+ < \infty, \quad a^- + p^- > 1, \] \hfill (3.3)

- the exponent \( y(z) = \frac{\alpha(z)}{p(z)-1} \) satisfies
  \[ |\nabla y(z)|^{p(z)} \in L^1(Q), \quad \partial_t y(z) \in L^2(Q), \] \hfill (3.4)

The solution of problem (3.1) is understood in the following sense.

**Definition 3.1.** A function \( u(z) \) is called weak solution of problem (3.1) if

1. \( u \in W(Q) \cap L^\infty(Q), \partial_t \Phi_0(z, v) \in W^1(Q), \)
2. for every \( \phi \in W(Q) \)
\[ \int_Q \left( \phi \partial_t \Phi_0(z, v) + b(z, v) |\nabla v + B(z, v)|^{p(z)-2} (\nabla v + B(z, v)) \cdot \nabla \phi - f \phi \right) dz = 0, \] \hfill (3.5)
Theorem 4.1. For every \( u \in \mathbb{C}^\infty_0 (\Omega) \), \( \int_\Omega \Phi_0 (\gamma (x)) \phi (x) dx \to \int_\Omega \Phi_0 (x, 0) \phi (x) dx \) as \( t \to 0 \).

The main result of the paper is given in the following theorem.

**Theorem 3.1.** Let the exponent \( p(z) \) satisfy the log-continuity condition conditions (2.1) in \( Q \), and let conditions (3.2), (3.3), (3.4) be fulfilled. Then for every \( f \in L^1 (0, T; L^\infty (\Omega)) \), \( u_0, v_0 \in L^\infty (\Omega) \) problem (3.1) has at least one weak solution \( v(z) \) in the sense of Definition 3.1.

4. Regularization of problem (3.1)

A solution of problem (3.1) has the form \( v = \Phi_0^{-1} (z, u) \) where \( u \) is obtained as the limit of the sequence of solutions of the regularized problems

\[
\begin{aligned}
\begin{cases}
\partial_t u = \text{div} \left( A_{\varepsilon, K}(z, u) |\nabla u|^{p(z)-2} \nabla u \right) + f(z) & \text{in } Q, \\
u(x, 0) = u_0 & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma,
\end{cases}
\end{aligned}
\]

(4.1)

with the coefficient \( A_{\varepsilon, K}(z, u) = a(z, u) (\varepsilon + |K|, |u|)^{p(z)} \), depending on the given parameters \( \varepsilon \in (0, 1), K > 0 \). For every \( \varepsilon \in (0, 1), 1 < K < \infty \) the coefficient \( A_{\varepsilon, K}(z, u) \) is separated away from zero and infinity which allows us to treat problem (4.1) as the Dirichlet problem for the evolutional \( p(z) \)-Laplacian.

**Definition 4.1.** ([10]). A function \( u \in L^\infty (0, T; L^2 (\Omega)) \cap W (Q) \) is called weak solution of problem (4.1) if for every test-function \( \phi \in L^\infty (0, T; L^2 (\Omega)) \cap W (Q) \) such that \( \phi_t \in W' (Q) \) and for every \( t_1, t_2 \in [0, T] \)

\[
\int_\Omega u \phi dx|_{t_1}^{t_2} = \int_1^{t_2} \int_\Omega |u \phi_t - A_{\varepsilon, K}(z, u) |\nabla u|^{p(z)-2} \nabla u \cdot \nabla \phi - f \phi| dz.
\]

(4.2)

**Theorem 4.1.** For every \( u_0 \in L^2 (\Omega), f \in L^2 (\Omega), \varepsilon > 0, K > 0 \) problem (4.1) has at least one weak solution \( u \in L^\infty (0, T; L^2 (\Omega)) \cap W (Q) \) such that \( u_t \in W' (Q) \). Moreover, if \( u_0 \in L^\infty (\Omega), f \in L^1 (0, T; L^\infty (\Omega)) \), then this solution belongs to \( L^\infty (Q) \) and obeys the estimate

\[
\|u\|_{L^\infty (Q)} \leq \|u_0\|_{L^\infty (\Omega)} + \int_0^T \| f(\cdot, s)\|_{L^\infty (\Omega)} ds \equiv K_0.
\]

(4.3)

**Sketch of proof.** The detailed proof of Theorem 4.1 can be found in [10]. The solution is thought as the limit of the sequence of Galerkin’s approximations \( u^{(m)} = \sum_{i=1}^m c_i (t) \psi_i (x) \), where \( \{ \psi_i \} \) is a basis of the space \( V_+ (\Omega) \). We may assume that the system \( \{ \psi_i \} \) is orthogonal in \( L^2 (\Omega) \). Substituting \( u^{(m)} \) into (4.2) with the test-functions \( \psi_i \) we obtain a system of ordinary differential equations for the coefficients \( c_i (t) \). This system can be solved on the interval \([0, T]\), and the functions \( u^{(m)} \) admit the uniform estimate

\[
\|u^{(m)}\|_{L^\infty (0, T; L^2 (\Omega))} + \int_Q |\nabla u^{(m)}|^{p(z)} dz \leq C \left( \|u_0\|^2_{L^2 (\Omega)} + \|f\|^2_{L^2 (Q)} \right),
\]

which allows one to pass to the limit in the sequence \( \{u^{(m)}\} \). The justification of the limit passage as \( m \to \infty \) is based on the monotonicity of the operator \( |s|^{p-2} s : \mathbb{R}^n \to \mathbb{R}^n \) and the formulas of integration by parts in \( t \) for the elements of the spaces \( W (Q), W'(Q) \) (see Lemma 2.5).

**Corollary 4.1.** According to (4.3), for \( u_0 \in L^\infty (\Omega), f \in L^1 (0, T; L^\infty (\Omega)) \) and \( K > K_0 \) the solutions of the regularized problem (4.1) do not depend on the parameter \( K \) and, in fact, are solutions of the problem

\[
\begin{aligned}
\begin{cases}
\partial_t u_{\varepsilon} = \text{div} \left( A_{\varepsilon, K}(z, u_{\varepsilon}) |\nabla u_{\varepsilon}|^{p(z)-2} \nabla u_{\varepsilon} \right) + f(z), & \text{for } z \in Q, \\
u_{\varepsilon}(x, 0) = u_0 & \text{in } \Omega, \\
u_{\varepsilon} = 0 & \text{on } \Gamma, \\
A_{\varepsilon, \infty}(z, u_{\varepsilon}) = a(z, u_{\varepsilon}) (\varepsilon + |u_{\varepsilon}|)^{p(z)(p(z)-1)}, & \varepsilon > 0, \gamma (z) = \frac{\alpha(z)}{p(z) - 1}.
\end{cases}
\end{aligned}
\]

(4.4)
Corollary 4.2. It follows from Theorem 4.1 and Lemma 2.5 that if a function $u \in L^\infty(0, T; L^2(\Omega)) \cap W(Q)$ is a weak solution of problem (4.1) in the sense of Definition 4.1, then $\partial_t u \in W'(Q)$ and for every test-function $\phi \in W(Q)$

$$
\int_Q \left[ \phi \partial_t u + A_{\epsilon, K}(z, u) |\nabla u|^{p(-) - 2} \nabla u \cdot \nabla \phi - f \phi \right] \, dz = 0.
$$

(4.5)

Taking for the test-function $\phi(x) \in C^\infty_0(\Omega)$ and applying Lemma 2.5 to the first term on the left-hand side of (4.5), we find that for every $t_1, t_2 \in [0, T]$

$$
\int_\Omega u \phi(x) \, dx |_{t_1}^{t_2} = \int_{t_1}^{t_2} \left[ A_{\epsilon, K}(z, u) |\nabla u|^{p(-) - 2} \nabla u \cdot \nabla \phi(x) - f \phi(x) \right] \, dz,
$$

which yields

$$
\lim_{t \to 0} \int_\Omega u \phi(x) \, dx = \int_\Omega u_0(x) \phi(x) \, dx \quad \forall \phi(x) \in C^\infty_0(\Omega)
$$

by virtue of the absolute continuity of the integral.

Remark 4.1. Existence of solutions to the evolutional $p(x, t)$-Laplace equation is proved in [1] under weaker assumptions on the exponent $p(z)$: it is assumed that $p(x, t)$ is log-continuous with respect to $t$ and $p(\cdot, t) \in C^0(\overline{\Omega})$. In this case a solution to the Dirichlet problem for the equation

$$
u_t = \text{div} A(\nabla u), \quad A(\nabla u) = |\nabla u|^{p(-) - 2} \nabla u : W(Q) \mapsto W'(Q),
$$

is obtained as the limit of the sequence of solutions of the regularized equations

$$
u_t = \text{div} A_{\epsilon}(\nabla u), \quad A_{\epsilon}(\nabla u) = \varepsilon |\nabla u|^{p(-) - 2} \nabla u + |\nabla u|^{p(-) - 2} \nabla u, \quad \varepsilon > 0.
$$

It is shown that

$$
L^{p(\cdot)}(0, T; V_+((\Omega) \in u_\epsilon \to u \in W(Q), \quad L^{p(\cdot)'}(0, T; V_+^*(\Omega)) \in A_{\epsilon}(\nabla u_\epsilon) \to A(\nabla u) \in W'(Q).
$$

Remark 4.2. This note does not address the question of uniqueness of solution to problem (3.1). Uniqueness of energy solutions to the stationary counterpart of equation (3.1) with variable exponents $p(x)$ and $\gamma(x)$ is proved in [11] by a method based on the ideas of paper [4]. In the case of constant exponents $\gamma, p$, and $\Phi(\cdot, v) \equiv \phi(v)$, uniqueness of solution to problem (3.1) is proved in [14,15] for the class of solutions satisfying $\partial_t \phi(v) \in L^1(\Omega)$ and, under less restrictive assumptions, in [26] (see also [20]). Uniqueness of energy solutions $v \in W(Q) \cap L^\infty(Q)$ for equation (3.1) with variable $p(z)$ and $\gamma(z)$ does not follow directly from the results of the cited works. The difficulty of this issue seems to be not merely technical, which is why we postpone its discussion for further publications.

4.1. A priori estimates

Due Corollary 4.1, from now on we may consider problem (4.1) as a problem with the unique regularization parameter $\varepsilon$. We devote this subsection to derive several independent of $\varepsilon$ a priori estimates for the solutions of problem (4.4). It is always assumed that the conditions of Theorem 4.1 are fulfilled. Let us introduce the vector-valued function $G_{\epsilon} = A_{\epsilon}(z, u_\epsilon)|\nabla u_\epsilon|^{p(-) - 2} \nabla u_\epsilon$ and write equation (4.4) in the form

$$
\partial_t u_\epsilon = \text{div} G_{\epsilon} + f(z).
$$

(4.6)

Lemma 4.1. The functions $G_{\epsilon}$ satisfy the estimate

$$
\int_Q |G_{\epsilon}|^{p'(z)} \, dz \leq C, \quad p'(z) = \frac{p(z)}{p(z) - 1},
$$

(4.7)

with an independent of $\varepsilon$ constant $C = C(a^+, p^-, K_0, \|f\|_{L_2(Q)}, \|\nabla \gamma\|_{L^p(\cdot, Q)}).$
Proof. Since \( u_\varepsilon \in \mathbf{W}(Q) \), then
\[
\phi_\varepsilon = \int_0^{u_\varepsilon} (\varepsilon + |s|)^\gamma(z) ds \in \mathbf{W}(Q) \quad \text{for every } \varepsilon > 0
\]
and can be taken for the test-function in (4.5) – see the proof of Lemma 6. Applying Lemma 6 to (4.5) with the test-function \( \phi_\varepsilon \), we arrive at the inequality
\[
a^{-1} \int_Q |G_\varepsilon|^p(z) dz \leq C + \int_Q |f| \phi_0 dz + \int_Q |G_\varepsilon| |\nabla \gamma'| \int_0^{u_\varepsilon} (\varepsilon + |s|)'^\gamma \ln(\varepsilon + |s|) ds dz
\]
in which the constant \( C \) depends only on \( K_0, \gamma^+, \|\gamma\|_{2,Q} \) and
\[
I_0 = \int_\Omega |u_0| \int_0^{u_0} (\varepsilon + |s|)^\gamma(x,0) ds.
\]
Estimating the last term on the right-hand side by Young’s inequality we obtain
\[
a^{-1} \int_Q |G_\varepsilon|^p(z) dz \leq C' \left( 1 + \int_Q |\nabla \gamma(z)|^p(z) dz \right)
\]
with a known constant \( C' \), depending on \( \gamma^+, K_0, p^+, \|\gamma\|_{2,Q}, \|\gamma\|_{p(\gamma),Q}, \|\gamma\|_{2,Q} \) and \( I_0 \).

Lemma 4.2. Under the conditions of Theorem 4.1 \( \|\partial_t u_\varepsilon\|_{\mathbf{W}(Q)} \leq C \) with a constant \( C \) independent of \( \varepsilon \).

Proof. It is sufficient to check that \( |(\partial_t u_\varepsilon, \phi)_{2,Q}| \leq C\|\phi\|_{\mathbf{W}(Q)} \) with a constant \( C \) independent of \( \phi \) and \( \varepsilon \) for every \( \phi \in \mathbf{W}(Q) \) such that \( \phi(x,0) = \phi(x, T) = 0 \). By virtue of (4.6)
\[
\left| \int_Q \phi \partial_t u_\varepsilon dz \right| \leq \int_Q |G_\varepsilon| |\nabla \phi| dz + \int_Q |f| \phi dz \leq \|G_\varepsilon\|_{p'(\gamma),Q} \|\nabla \phi\|_{p(\gamma),Q} + \|f\|_{2,Q} \|\phi\|_{2,Q}
\]
\[
\leq C \left( \|G_\varepsilon\|_{p'(\gamma),Q} + \|f\|_{2,Q} \right) \|\phi\|_{\mathbf{W}(Q)}
\]
and the needed estimate follows from Lemma 4.1.

Lemma 4.3. Under the foregoing assumptions the sequence of solutions of problem (4.4) is relatively compact in \( L^s(Q) \) with some \( 1 < s < \infty \).

Proof. Since the solutions \( u_\varepsilon \) are uniformly bounded, estimate (4.7) entails the inequality
\[
\int_Q |\nabla \left[ |u_\varepsilon|^{\gamma-1} u_\varepsilon \right]|^p dz \leq C_1 \int_Q (\varepsilon + |u_\varepsilon|)^{(\gamma-1)p} |\nabla u_\varepsilon|^{p} dz
\]
\[
\leq C_2 \int_Q (\varepsilon + |u_\varepsilon|)^{\gamma(z)p(z)} |\nabla u_\varepsilon|^{p(z)} dz + C_3 \int_Q (\varepsilon + |u_\varepsilon|)^{(\gamma-1)\gamma - p}|z|^{p(z)} dz \leq C_4,
\]
provided that \( \gamma \geq 1 + \gamma^+ \). The uniform estimates
\[
\|\partial_t u_\varepsilon\|_{\mathbf{W}(Q)} \leq C, \quad \|\nabla \left[ |u_\varepsilon|^{\gamma-1} u_\varepsilon \right] \|_{p(\gamma),Q} \leq C,
\]
and the results of [32, Sec. 8] yield relative compactness of the sequence \( \{u_\varepsilon\} \) in \( L^s(Q) \) with some \( s \in (1, \infty) \).

Gathering the above estimates we may find a function \( u \in L^\infty(Q) \cap L^s(Q) \) and a subsequence of \( \{u_\varepsilon\} \) (we conserve for this subsequence the same notation \( u_\varepsilon \)) such that
\[
\begin{align*}
\|u_\varepsilon\|_{\infty,Q} &\leq K_0, \quad K_0 = \text{const independent of } \varepsilon, \\
u_\varepsilon &\rightarrow u \text{ in } L^s(Q) \quad (1 < s < \infty), \\
u_\varepsilon &\rightarrow u \quad \text{a.e. in } Q, \\
\partial_t u_\varepsilon &\rightarrow \partial_t u \quad \text{weakly in } \mathbf{W}'(Q).
\end{align*}
\]
Let us introduce the functions
\[ v_\varepsilon(z) = \int_0^{u_\varepsilon(z)} (\varepsilon + |s|)^\gamma(z) ds, \quad \gamma(z) = \frac{\alpha(z)}{p(z) - 1} \geq \gamma^- > -1. \] (4.10)

Since the mapping \( u_\varepsilon \mapsto v_\varepsilon \) is monotone, there exists the inverse function \( u_\varepsilon = \Phi_\varepsilon(z, v_\varepsilon) \). The following formulas hold:
\[
\begin{align*}
(a) & \quad \nabla v_\varepsilon = (\varepsilon + |u_\varepsilon|)^\gamma \nabla u_\varepsilon + \nabla \gamma \int_0^{u_\varepsilon} (\varepsilon + |s|)^\gamma \ln(\varepsilon + s) ds, \\
(b) & \quad \partial_t v_\varepsilon = (\varepsilon + |u_\varepsilon|)^\gamma \partial_t u_\varepsilon + \gamma \int_0^{u_\varepsilon} (\varepsilon + |s|)^\gamma \ln(\varepsilon + s) ds.
\end{align*}
\] (4.11)

By the assumption \( \gamma^- > -1 \), due to Lemma 4.1, uniform in \( \varepsilon \) boundedness of \( u_\varepsilon \) yields
\[ |\nabla v_\varepsilon|^{p(z)} \leq C(K_0) \left[ |G_\varepsilon|^{p(z)} + |\nabla \gamma|^{p(z)} \right] \in L^1(Q), \]
which gives the estimate \( ||\nabla v_\varepsilon||_{p(z), Q} \leq C \) with an independent of \( \varepsilon \) constant \( C \). It follows that there exist a subsequence of \( \{v_\varepsilon\} \) and a function \( v \in L^{p(z)}(Q) \) such that
\[ |\nabla v_\varepsilon|^{p(z)-2} \nabla v_\varepsilon \to V \quad \text{weakly in } L^{p(z)}(Q). \] (4.12)

On the other hand, the continuity of the mapping \( u_\varepsilon \mapsto v_\varepsilon \) and the convergence \( u_\varepsilon \to u \) (see (4.9)) give
\[ \begin{align*}
|v_\varepsilon| & \leq C(K_0), \\
v_\varepsilon & \to v \quad \text{in } L^q(Q) \text{ for some } q \in (1, \infty), \\
v_\varepsilon & \to v \quad a.e. \text{ in } Q, \\
\nabla v_\varepsilon & \to \nabla v \quad \text{weakly in } L^{p(z)}(Q),
\end{align*} \] (4.13)
with
\[ v = \int_0^u |s|^{\gamma} ds = \frac{u|u|^\gamma}{\gamma + 1} \in W(Q). \]

The next step is to check that the limit function \( u = \lim_{\varepsilon \to 0} \Phi_\varepsilon(z, v_\varepsilon) \) is a solution of problem (3.1).

5. Existence of weak solution to problem (3.1). Proof of Theorem 3.1

Let us write the regularized problems (4.4) as the problem for the functions \( v_\varepsilon \) defined by (4.10):
\[ \begin{align*}
\partial_t \Phi_\varepsilon(z, v_\varepsilon) & = d_i v \left( b(z, v_\varepsilon)[\nabla v_\varepsilon + B(z, v_\varepsilon)]^{p(z)-2}(\nabla v_\varepsilon + B(z, v_\varepsilon)) + f \right) \text{ in } Q, \\
v_\varepsilon & = 0 \text{ on } \Gamma, \quad v_\varepsilon(x, 0) = v_0(x) \text{ in } \Omega,
\end{align*} \] (5.1)
where
\[ B(z, v_\varepsilon) = -\nabla \gamma(z) \int_0^{u_\varepsilon} (\varepsilon + |s|)^\gamma \ln(\varepsilon + s) ds, \quad b(z, v_\varepsilon) \equiv a(z, u_\varepsilon), \quad u_\varepsilon = \Phi_\varepsilon(z, v_\varepsilon). \]

For every \( \varepsilon > 0 \) problem (4.4) has at least one bounded solution \( u_\varepsilon = \Phi_\varepsilon(z, v_\varepsilon) \in W(Q) \). The corresponding function \( v_\varepsilon = \Phi^{-1}_\varepsilon(z, u_\varepsilon) \) is a solution of problem (5.1) in the sense of Definition 3.1. By virtue of (4.13) and (4.9) there exist functions \( u \in L^2(\Omega) \) and \( A \in L^{p/z}(\Omega) \) such that for a subsequence of solutions of problem (5.1) \( \{v_\varepsilon\} \) (for this subsequence we conserve the same notation \( \{v_\varepsilon\}\))
\[ \begin{align*}
v_\varepsilon & \to v \equiv \frac{1}{\gamma + 1} |u|^\gamma u, \quad \Phi_\varepsilon(z, v_\varepsilon) \to \Phi_0(z, v) \equiv u, \\
B(z, v_\varepsilon) & \to B(z, v) = \frac{\nabla \gamma}{\gamma + 1} v (1 + (1 + \gamma) \ln |v|) = \frac{\nabla \gamma}{\gamma + 1} \left( \frac{u|u|^\gamma}{\gamma + 1} - u|u|^\gamma \ln |u| \right) \text{ a.e. in } Q, \\
b(z, v_\varepsilon)[\nabla v_\varepsilon + B(z, v_\varepsilon)]^{p(z)-2}(\nabla v_\varepsilon + B(z, v_\varepsilon)) & \to A \quad \text{weakly in } L^{p/z}(Q).
\end{align*} \] (5.2)
Letting $\varepsilon \to 0$ in (3.5) for $u_\varepsilon$, we find that for every $\phi \in W(Q)$,
\begin{align}
\int_Q (u_\varepsilon \phi + \Lambda \nabla \phi - f \phi) = 0, \quad u = \Phi_0(z, v).
\end{align}

To complete the proof of existence amounts to check that
\begin{align}
\forall \phi \in W(Q) \quad \int_Q A(z) \cdot \nabla \phi dz = \int_Q b(z, v)|\nabla v + B(z, v)|^{p(z)-2}(\nabla v + B(z, v)) \cdot \nabla \phi dz.
\end{align}

Let us define the function $F(\xi, \eta) = |\nabla \xi + B(z, \eta)|^{p(z)-2}(\nabla \xi + B(z, \eta))$ and make use of the representation
\begin{align}
F(v_\varepsilon, v_\varepsilon) - F(v, v) = |\nabla v_\varepsilon + B(z, v_\varepsilon)|^{p(z)-2}(\nabla v_\varepsilon + B(z, v_\varepsilon)) - |\nabla v + B(z, v)|^{p(z)-2}(\nabla v + B(z, v))
\end{align}
\begin{align*}
= [F(v_\varepsilon, v_\varepsilon) - F(v_\varepsilon, v)] + [(F(v_\varepsilon, v) - F(v, v)] \equiv J_1^{(\varepsilon)} + J_2^{(\varepsilon)}
\end{align*}

5.1. Step 1: $J_1^{(\varepsilon)} \to 0$ as $\varepsilon \to 0$

Let us assume that $p(z) \geq 2$ at a point $z \in Q$. According to Proposition 2.1 with $a = \nabla v_\varepsilon + B(z, v_\varepsilon)$, $b = \nabla v_\varepsilon + B(z, v)$ and $p = p(z)$
\begin{align}
|J_1^{(\varepsilon)}| \leq C(p^+, p^-)|B(z, v_\varepsilon) - B(z, v)| \left(|B(z, v_\varepsilon)|^{p(z)-1} + |B(v)|^{p(z)-1} + |\nabla v_\varepsilon|^{p(z)-1}\right).
\end{align}

If $p(z) \in (1, 2)$, we apply Proposition 2.2 to obtain
\begin{align}
|J_1^{(\varepsilon)}| \leq C(p^+, p^-)|B(z, v_\varepsilon) - B(z, v)|^{1-\alpha \left(|B(z, v_\varepsilon)|^{p(z)-2+\alpha} + |B(z, v)|^{p(z)-2+\alpha} + |\nabla v_\varepsilon|^{p(z)-2+\alpha}\right)}.
\end{align}

Set $q^+(z) = \min\{p(z), 2\}$, $q^-(z) = \min\{p(z), 2\}$, $Q^+ = Q \cap \{z: p(z) \geq 2\}$, $Q^- = Q \cap \{z: p(z) \in (1, 2)\}$. For every test-function $\phi \in W(Q)$
\begin{align}
\left|\int_Q J_1^{(\varepsilon)} \cdot \nabla \phi dz\right| \leq \int_Q \phi^{1-\beta:p}(z) dz
\end{align}
\begin{align}
\leq C_1(p^+) \int_{Q^+} |B(z, v_\varepsilon) - B(z, v)| \left(|B(z, v_\varepsilon)|^{p(z)-1} + |B(v)|^{p(z)-1} + |\nabla v_\varepsilon|^{p(z)-1}\right) dz
\end{align}
\begin{align}
+ C_2(p^+) \int_{Q^-} |B(z, v_\varepsilon) - B(z, v)|^{1-\beta \left(|B(z, v_\varepsilon)|^{p(z)-2+\beta} + |B(z, v)|^{p(z)-2+\beta} + |\nabla v_\varepsilon|^{p(z)-2+\beta}\right)} dz
\end{align}
with $2 - p < \beta < 1$. Applying Hölder’s inequality we conclude that the right-hand side of this inequality tends to zero as $\varepsilon \to 0$.

5.2. Step 2: $J_2^{(\varepsilon)} \to F(v, v)$ as $\varepsilon \to 0$

Taking $v_\varepsilon$ for the test-function in identity (3.5) for $u_\varepsilon$ and applying the integration-by-parts formula (2.8) we obtain the energy relation
\begin{align}
\mu_\varepsilon(u_\varepsilon, v_\varepsilon) + \int_Q b(z, v_\varepsilon)F(v_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon = \int_Q f v_\varepsilon dz
\end{align}
with $\mu_\varepsilon(\cdot, \cdot)$ defined in (2.8). Let us recall that $|\mu_\varepsilon|$ are bounded uniformly with respect to $\varepsilon$.

Lemma 5.1. For every $\forall \phi \in W(Q)$
\begin{align}
\lim_{\varepsilon \to 0} \int_{Q_T} b(z, v_\varepsilon)F(v_\varepsilon, v) \cdot \nabla \phi dz = \int_{Q_T} b(z, v)F(v, v) \cdot \nabla \phi dz.
\end{align}
Proof. The proof is an adaptation of the arguments of [24, Chapter 2, Section 1.2.2]. By monotonicity

\[ 0 \leq \int_Q b(z, v_\varepsilon) (F(v_\varepsilon, v) - F(\eta, v)) \cdot \nabla(v_\varepsilon - \eta) \, dz. \]  

(5.7)

Subtracting (5.5) from (5.7), we get

\[ 0 \leq -\mu_\varepsilon(u_\varepsilon, v_\varepsilon) - \int_Q b(z, v_\varepsilon) F(v_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon \, dz 
+ \int_Q b(z, v_\varepsilon)(F(v_\varepsilon, v) - F(\eta, v)) \cdot \nabla(v_\varepsilon - \eta) \, dz 
+ \int_Q f v_\varepsilon \, dz \]

(5.8)

where

\[
I_1(\varepsilon) = \int_Q b(z, v_\varepsilon)(F(v_\varepsilon, v) - F(v_\varepsilon, v_\varepsilon)) \cdot \nabla v_\varepsilon \, dz,
\]

\[
I_2(\varepsilon) = -\int_Q b(z, v_\varepsilon) F(\eta, v) \cdot \nabla v_\varepsilon \, dz,
\]

\[
I_3(\varepsilon) = -\int_Q b(z, v_\varepsilon) F(v_\varepsilon, v) \cdot \nabla \eta \, dz,
\]

\[
I_4(\varepsilon) = \int_Q b(z, v_\varepsilon) F(\eta, v) \cdot \nabla \eta \, dz,
\]

\[
I_5(\varepsilon) = \int_Q f v_\varepsilon \, dz.
\]

Using the continuity of \(b(z, s)\) with respect to \(s\) and the weak convergence

\[ \nabla v_\varepsilon \to \nabla v, \quad b(z, v_\varepsilon) F(v_\varepsilon, v_\varepsilon) = (b(z, v_\varepsilon) - b(z, v))F(v_\varepsilon, v_\varepsilon) + b(z, v)F(v_\varepsilon, v_\varepsilon) \to \Lambda, \]

it is easy to see that

\[
\lim I_2(\varepsilon) = -\int_Q b(z, v) F(\eta, v) \cdot \nabla v \, dz,
\]

\[
\lim I_3(\varepsilon) = -\int_Q \Lambda \cdot \nabla \eta \, dz,
\]

\[
\lim I_4(\varepsilon) = \int_Q b(z, v) F(\eta, v) \cdot \nabla \eta \, dz.
\]

Let us consider the term \(I_1(\varepsilon)\). Since

\[ \mathcal{B}(z, v_\varepsilon) \to \mathcal{A}(z, v) \text{ a.e. in } Q, \text{ } \int_Q |\nabla v_\varepsilon|^p(v(z)) \, dz \leq C, \]

we find, repeating the proof of (5.4), that \(|I_1| \to 0\) as \(\varepsilon \to 0\). Letting \(\varepsilon \to 0\) in (5.8) and using the inequality

\[ \liminf_{\varepsilon \to 0} \mu_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \mu(u, v) = \int_Q u_\varepsilon v_\varepsilon \, dz \]

with \(\mu\) given in (2.10), we obtain the inequality

\[
0 \leq -\mu(u, v) - \int_Q b(z, v) F(v, v) \cdot \nabla v \, dz - \int_Q \Lambda \cdot \nabla \eta \, dz 
+ \int_Q b(z, v) F(\eta, v) \cdot \nabla \eta \, dz + \int_Q f v \, dz. \]

(5.9)
Combining (5.9) with (2.10) we have

\[ 0 \leq \int_Q (\Lambda - b(z, v)F(\eta, v) \cdot \nabla (v - \eta)) \, dz. \]  
(5.10)

Let us take \( \eta = v - \lambda w \) with \( \lambda = \text{const} \) and \( w \in \mathcal{W}(Q) \). Under this choice of the test-function

\[ 0 \leq \lambda \int_Q (\Lambda - b(z, v)F(v - \lambda w, v)) \cdot \nabla w \, dz. \]  
(5.11)

Simplifying and letting \( \lambda \to 0 \) we find that

\[ 0 \leq \int_Q (\Lambda - b(z, v)F(v, v)) \cdot \nabla w \, dz. \]

Since \( w \in \mathcal{W}(Q) \) is arbitrary, it follows that \( \Lambda = b(z, v)F(v, v) \).

The proof of Theorem 3.1 is completed.

References


