On the evolution $p$-Laplacian with nonlocal memory

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We study the homogeneous Dirichlet problem for the evolution $p$-Laplacian with the nonlocal memory term

$$
\begin{align*}
    u_t - \Delta_p u &= \int_0^t g(t-s) \Delta_p u(x,s) \, ds \\
    &\quad + \Theta(x,t,u) + f(x,t) \quad \text{in } Q = \Omega \times (0,T),
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\Theta$, $g$ and $f$ are given functions. It is proved that for $\max \left\{ 1, \frac{n}{2+\frac{n}{p}} \right\} < p < \infty$, $g, g' \in L^2(0,T)$ and $u_0 \in W^{1,p}_0(\Omega)$, $f \in L^2(Q)$ the problem admits a weak solution, which is global or local in time in dependence on the growth rate of $\Theta(x,t,s)$ as $|s| \to \infty$. Conditions of uniqueness are established. It is proved that for $p > 2$ and $s \Theta(x,t,s) \leq 0$ the disturbances from the data propagate with finite speed and the “waiting time” effect is possible. We present simple explicit solutions that show the failure of the maximum and comparison principles for the solutions of equation (0.1).

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1. Introduction

We study the Dirichlet problem for the evolution $p$-Laplace equation with the nonlocal term

$$
\begin{align*}
    u_t - \Delta_p u &= \int_0^t g(t-s) \Delta_p u(x,s) \, ds + \Theta(x,t,u) + f(x,t) \quad \text{in } Q = \Omega \times (0,T), \\
    u &= 0 \quad \text{on } \partial \Omega \times [0,T], \\
    u(x,0) &= u_0(x) \quad \text{in } \Omega
\end{align*}
$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz-continuous boundary, $g(s)$ is a given memory kernel, $\theta$ is a given function and $\Delta_p$ denotes the $p$-Laplacian

$$\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad 1 < p < \infty.$$ 

Eq. (1.1) appears in the mathematical description of the heat propagation in materials with memory where the heat flux may depend on the past history of the process. Let $u(x,t)$ denote the temperature at the point $(x,t)$, $\epsilon(x,t)$ be the internal energy and $q(x,t)$ be the heat flux. Following [15] we may assume that the internal energy is proportional to the temperature, $\epsilon(x,t) = bu(x,t)$ with $b = \text{const} > 0$, and that the dependence of heat flux $q$ on the temperature gradient $\nabla u$ is nonlinear, nonlocal in time, and is defined by the functional

$$q(x,t) = -\sigma(\nabla u(x,t)) - \int_0^t g(t-s)\sigma(\nabla u(x,s)) \, ds$$

with given functions $\sigma$ and $g$. Assume also that the external heat supply may depend on the temperature and is defined by a function $\Phi(x,t,u)$. Under these assumptions the energy balance law

$$\epsilon_t + \text{div} q = \Phi$$

transforms into the equation

$$bu_t(x,t) - \text{div}(\nabla u(x,t)) = \int_0^t g(t-s)\text{div}(\nabla u(x,s)) \, ds + \Phi(x,t,u).$$

Scaling out the constant $b$ and assuming $\sigma(s) = |s|^{p-2}s$, $s \in \mathbb{R}^n$, we arrive at Eq. (1.1).

In the recent decades, the evolution equations with memory terms have been intensively studied. The questions of solvability and the long time behavior of solutions of the nonlinear Volterra equation

$$u_t(t) - Bu(t) + \int_0^t g(t-s)Au(s) \, ds = f(t)$$

and nonlocal equations of similar structure were studied in [5,10,20,23]. In [10,23] the solution $u(t)$ takes values in a reflexive Banach space $W$, $u_t$ is an element of the dual space $W'$, and $A, B$ are the subdifferentials of convex, lower semicontinuous and proper functions. An example of such an equation is furnished by

$$u_t(x,t) - u_{xx}(x,t) = \int_0^t g(t-s)(\sigma(u_x(x,s)))_x \, ds = f(x,t)$$

with a sufficiently smooth function $\sigma$—see [10, p. 727]. For the semilinear equation with $p = 2$ and $\theta \neq 0$ the same questions were addressed by many authors, see, e.g., [6–8] and references therein. Nonexistence of global solutions (a finite time blow-up) for semilinear equations was studied in [19,21,22]. Similar results are also known for nonlocal parabolic equations and boundary conditions of other types, see, e.g., [12,13,16,17]. Doubly nonlinear nonlocal parabolic equations

$$\partial_t \beta(u) - \text{div}(\nabla u) = \int_0^t g(t-s)\text{div}(\nabla u(s)) \, ds + f(x,t,u)$$

were studied in [14,24,26,27] in an abstract setting. The results concerning the global existence apply to Eq. (1.1) if $\beta(s) = s$ and the forcing term $f(x,t,s)$ is Lipschitz-continuous with respect to $s$. The method of proof is based on the time-discretization scheme proposed in [26].

The aim of the present work is to derive sufficient conditions of global and local in time existence of solutions to problem (1.1) under less restrictive assumptions on $\theta$ and to study the property of finite speed of propagation of disturbances from the data. Let us denote $W = L^2(Q) \cap L^p(0,T;W^{1,p}_0(\Omega))$. 
Definition 1.1. A function \( u(x,t) \) is called weak solution of problem (1.1) if

1. \( u \in C([0,T]; L^2(\Omega)) \cap W, \ u_t \in L^2(Q), \ \Delta_p u \in L^2(Q), \ u(x,0) = u_0(x) \),
2. for every test-function \( \phi \in W \)

\[
\int_Q \left( u_t \phi + \nabla \phi \cdot |\nabla u|^{p-2} \nabla u + \int_0^t g(t-s)|\nabla u(s)|^{p-2} \nabla u(s) ds \right) dx dt = \int_Q \left( f + \Theta(x,t,u) \right) \phi dx dt.
\]

It is worth noting that for \( f \in L^2(Q) \) Eq. (1.1) is fulfilled almost everywhere in \( Q \).

A customary approach to the study of equations that involve \( p \)-Laplacian is based on monotonicity (see, e.g., [18, Ch.2]), but the lack of monotonicity in the nonlocal term prevents one from a direct application of this well-developed technique. In order to overcome this specific difficulty we substitute Eq. (1.1) by the system composed of the diffusion-reaction equation and the integral equation of Volterra type

\[
\begin{aligned}
\begin{cases}
    u_t - \Delta_p u = Y + \Theta(x,t,u) + f & \text{in } Q, \\
    u = 0 \text{ on } \partial \Omega \times [0,T], \\
    u(x,0) = u_0(x) \text{ in } \Omega,
\end{cases}
\end{aligned}
\]

(1.2)

\[
Y(t) = \int_0^t g(t-s) Y(s) ds + F(x,t,u).
\]

(1.3)

It turns out that for a suitably chosen \( F \) the solution of system (1.2)–(1.3) generates a solution of problem (1.1).

The paper is organized as follows. We begin with establishing several properties of the Volterra equation (1.3) (Section 2). Although this class of equations is exhaustively studied, (see, e.g., [9]), we include these results for the sake of convenience of presentation.

The existence theorem is proved in Section 3. We construct two different sequences of finite-dimensional Galerkin’s approximations for the solutions of Eqs. (1.2), (1.3) \( u \) and \( Y \), whose coefficients are defined from a weakly coupled system of nonlinear ODEs, and then show that each of the two sequences has a limit. Gathering these limits we obtain a solution of the original problem (1.1). The main result is given in Theorem 3.1: for every \( u_0 \in W_0^{1,p}(\Omega), \ f \in L^2(Q), \ p > \frac{2n}{n+2} \) and \( g, g' \in L^2(0,T), \ |g(0)| < \infty \) problem (1.1) has at least one weak solution, which is global or local in time in dependence on the growth rate of \( \Theta(x,t,s) \) as \( |s| \to \infty \). The existence result do not require continuity of \( \Theta(x,t,s) \) in \( s \).

Uniqueness of weak solutions is established under the stronger assumption of Lipschitz-continuity of \( \Theta(x,t,s) \) with respect to \( s \) (Section 4).

In Section 5 we study the property of finite speed of propagation for the solutions of problem (1.1) under the assumptions \( p > 2 \) and \( s\Theta(x,t,s) \leq 0 \). It is shown that if \( u_0 = 0 \) in \( B_{\rho_0}(x_0) = \{ x : |x - x_0| < \rho_0 \} \subset \Omega \) and \( f = 0 \) in \( B_{\rho_0} \times (0,T) \), then for small times \( u(x,t) = 0 \ a.e. \) in a ball \( B_{\rho(t)}(x_0) \) of variable decreasing radius \( \rho(t) \). Moreover, under certain conditions on the rate of vanishing of \( u_0 \) and \( f \) near the boundaries of their supports the solutions of Eq. (1.1) may display the waiting time phenomenon, which means that \( u(x,t) = 0 \ a.e. \) in \( B_{\rho_0}(x_0) \times (0,t_s) \) for some \( t_s \) depending on the data, that is, parts of the boundary of supp \( u(x,t) \) remain stable for small times. The analysis of the localization properties is practically independent of the existence theorem and does not require reduction of the nonlocal equation (1.1) to system (1.2)–(1.3). Our approach is based on the method of local energy estimates developed in the monograph [2], see also papers [1,3,4].
The possibility of localization of solutions in space or time is an intrinsic property of the nonlinear diffusion equation

\[
\begin{aligned}
&\begin{cases}
  u_t = \Delta_p u & \text{in } Q, \ p \neq 2, \\
  u = 0 & \text{on } \partial\Omega \times (0, T), \\
  u(x, 0) = u_0(x) & \text{in } \Omega.
  \end{cases}
\end{aligned}
\] (1.4)

It is well-known that the solutions of problem (1.4) vanish in a finite time if \( p \in (1, 2) \), or possess the property of finite speed of propagation if \( p > 2 \), an overview of the pertinent results can be found, e.g., in \([1,2,4]\). Although the latter property is preserved for the solutions of the nonlocal equation (1.1), the former one is not.

Let us assume \( F \in L^2(Q) \), \( g \in L^2(0, T) \) and consider Eq. (1.3) as the functional equation \( Y = \mathcal{M}(Y) \) with the operator

\[
\mathcal{M}(Z) = \int_0^t g(t - s)Z(s) \, ds + F.
\]

**Lemma 2.1.** \( \mathcal{M} : L^2(Q) \mapsto L^2(Q) \).

**Proof.** For every \( v \in L^2(Q) \)

\[
\|\mathcal{M}(v)\|_{2, Q}^2 \leq 2 \left( \int_0^t g(t - s)v(x) \, ds \right)^2_{2, Q} + 2\|F\|_{2, Q}^2
\]

\[
\leq 2\|g\|_{2, (0, T)} \int_Q \left( \int_0^t v^2(s) \, ds \right) \, dx dt + 2\|F\|_{2, Q}^2 \leq 2T\|g\|_{2, (0, T)}^2\|v\|_{2, Q}^2 + 2\|F\|_{2, Q}^2. \quad \square
\]

Let us denote \( Q_\theta = \Omega \times (0, \theta), \, \theta \in (0, T) \).

**Lemma 2.2.** The mapping \( \mathcal{M} : L^2(Q_\theta) \mapsto L^2(Q_\theta) \) is a contraction, provided that \( \theta\|g\|_{2, (0, T)}^2 < 1 \).

**Proof.** For every \( u, v \in L^2(Q) \)

\[
\|\mathcal{M}(u) - \mathcal{M}(v)\|_{2, Q_\theta}^2 = \left( \int_0^t g(t - s)(u - v)(s) \, ds \right)^2_{2, Q_\theta} \leq \theta\|g\|_{2, (0, T)}^2\|u - v\|_{2, Q_\theta}^2. \quad \square
\]

The next assertion immediately follows from Lemma 2.2 and the fixed point theorem for contraction mappings.

**Lemma 2.3.** Let \( g \in L^2(0, T) \) and \( \theta\|g\|_{2, (0, T)}^2 < 1 \). Then for every \( F \in L^2(Q) \) Eq. (1.3) has a unique solution \( Y \in L^2(Q_\theta) \).
**Lemma 2.4.** The solutions of Eq. (1.3) satisfy the estimates
\[
\|Y\|_{2,Q}^2 \leq 4e^{2T\|g\|_{2,(0,T)}^2} \|F\|_{2,Q}^2
\] (2.1)
and
\[
\|Y(t)\|_{2,\Omega}^2 \leq 4 \left( 2\|g\|_{2,(0,T)}^2 e^{2T\|g\|_{2,(0,T)}^2} \|F\|_{2,Q}^2 + \|F(t)\|_{2,\Omega}^2 \right) \quad \text{a.e. in } (0,T). \tag{2.2}
\]

**Proof.** Let \(Y \in L^2(Q)\) be a solution of (1.3) in \(Q\). Multiplying (1.3) by \(Y\), integrating over \(\Omega\) and applying the Cauchy inequality we find that for a.e. \(t \in (0,T)\)
\[
\|Y(t)\|_{2,\Omega}^2 \leq \int_0^t |Y(t)| \int_0^t |g(t-s)||Y(s)| \, ds \, dx + \int_\Omega |Y(t)||F(t)| \, dx
\]
\[
\leq \frac{1}{2}\|Y(t)\|_{2,\Omega}^2 + \frac{1}{2} \int_0^t \left( \int_0^t |g(t-s)||Y(s)| \, ds \right)^2 \, dx + \|F(t)\|_{2,\Omega} \|Y(t)\|_{2,\Omega}
\]
\[
\leq \frac{1}{2}\|Y(t)\|_{2,\Omega}^2 + \frac{1}{2}\|g\|_{2,(0,T)}^2 \int_0^t \|Y(s)\|_{2,\Omega}^2 \, ds + \|F(t)\|_{2,\Omega} \|Y(t)\|_{2,\Omega},
\]
Simplifying we arrive at integral inequality
\[
\|Y(t)\|_{2,\Omega}^2 \leq \|g\|_{2,(0,T)}^2 \int_0^t \|Y(s)\|_{2,\Omega}^2 \, ds + 2\|F(t)\|_{2,\Omega} \|Y(t)\|_{2,\Omega}
\]
\[
\leq \|g\|_{2,(0,T)}^2 \int_0^t \|Y(s)\|_{2,\Omega}^2 \, ds + \frac{1}{2}\|Y(t)\|_{2,\Omega}^2 + 2\|F(t)\|_{2,\Omega}^2,
\]
whence
\[
\|Y(t)\|_{2,\Omega}^2 \leq 2\|g\|_{2,(0,T)}^2 \int_0^t \|Y(s)\|_{2,\Omega}^2 \, ds + 4\|F(t)\|_{2,\Omega}^2 \quad \text{for a.e. } t \in (0,T). \tag{2.3}
\]
Since the function \(Z(t) = \int_0^t \|Y(s)\|_{2,\Omega}^2 \, ds\) satisfies the conditions
\[
Z'(t) \leq 2\|g\|_{2,(0,T)}^2 Z(t) + 4\|F(t)\|_{2,\Omega}^2, \quad Z(0) = 0,
\]
estimate (2.1) follows from Gronwall’s lemma, inequality (2.2) follows then by virtue of (2.3). \(\square\)

**Lemma 2.5.** Let \(g \in L^2(0,T)\). For every \(T > 0\) and every \(F \in L^2(Q)\) Eq. (1.3) has a unique solution \(Y \in L^2(Q)\).

**Proof.** It is shown in Lemma 2.3 that Eq. (1.3) has a unique solution in the cylinder \(Q_\theta\) of height \(\theta\) depending only on \(\|g\|_{2,(0,T)}\). Let us consider Eq. (1.3) in the cylinder \(Q_{2\theta} \setminus Q_\theta\):
\[
Y(t) = \int_\theta^t g(t-s)Y(s) \, ds + F^*, \quad t \in (\theta, 2\theta), \quad F^* = F + \int_0^\theta g(t-s)Y(s) \, ds. \tag{2.4}
\]
By Lemma 2.4
\[
\|F^*\|_{2,Q_{2\theta} \setminus Q_\theta}^2 \leq 2\|F\|_{2,Q}^2 + 2\|g\|_{2,(0,T)}^2 \|Y\|_{2,\Omega_\theta}^2 \leq C \|F\|_{2,Q}^2
\]
with a finite constant \(C = C(T, \|g\|_{2,(0,T)})\). By Lemma 2.3 Eq. (2.4) has a unique solution \(Y \in L^2(Q_{2\theta} \setminus Q_\theta)\). Continuing this process, in a finite number of steps \(k \leq T/\theta\) we exhaust the interval \((0,T)\) and obtain the solution of (1.3)
\[
Y \in L^2 \left( Q_\theta \cup (Q_{2\theta} \setminus Q_\theta) \cup \cdots \cup (Q_T \setminus Q_{(k-1)\theta}) \right). \quad \square
\]
Lemma 2.6. The solutions of Eq. (1.3) continuously depend on $F$. The solution of Eq. (1.3) is unique.

Proof. Let $Y_1, Y_2$ be the solutions of Eq. (1.3) corresponding to the right-hand sides $F_1, F_2$. Set $Y = Y_1 - Y_2$ and $F = F_1 - F_2$. The function $Y$ satisfies the equation

$$Y(t) = \int_0^t g(t-s)Y(s)ds + F.$$ 

Using (2.1) we find that

$$\|Y_1 - Y_2\|_{2,Q} \leq 4e^{2T\|g\|_{L^2(0,T)}}\|F_1 - F_2\|_{2,Q}.$$ 

Uniqueness is an immediate byproduct of this inequality. $\square$

3. Existence of weak solutions

Let us consider the problem

$$u_t - \Delta_p u = \int_0^t g(t-s)\Delta_p u(s)ds + \Theta(x,t,u) + f(x,t) \quad \text{in } Q = \Omega \times (0,T),$$

$$u = 0 \quad \text{on } \partial \Omega \times [0,T], \quad u(x,0) = u_0(x) \quad \text{in } \Omega. \quad (3.1)$$

It is assumed that

$$g, g' \in L^2(0,T), \quad f \in L^2(Q), \quad u_0 \in L^2(\Omega) \cap W^{1,p}_0(\Omega), \quad (3.2)$$

$$|\Theta(x,t,r)| \leq B |r|^{\sigma-1} \text{ with } \sigma \in (1,\infty), \quad B = \text{const} > 0. \quad (3.3)$$

3.1. Auxiliary problem

To construct a solution of the nonlocal problem (3.1) we consider the auxiliary problem of finding the pair $(u, Y)$ from the conditions

$$\begin{cases}
u_t - \Delta_p u = Y + \Theta(x,t,u) + f(x,t) & \text{in } Q, \\
u = 0 & \text{on } \partial \Omega \times [0,T], \\
u(x,0) = u_0(x) & \text{in } \Omega, \
\end{cases} \quad (3.4)$$

$$Y(t) = \int_0^t g(t-s)Y(s)ds + F(x,t,u), \quad (3.5)$$

where $F$ is the nonlinear nonlocal operator

$$F(x,t,u(x,\cdot)) = u(x,t)g(0) - u_0(x)g(t) - \int_0^t g'(t-s)u(x,s)ds + \int_0^t g(t-s)(f(x,s) + \Theta(x,s,u(x,s)))ds. \quad (3.6)$$

3.1.1. Galerkin’s approximations

A solution of problem (3.4)–(3.5) is obtained as the limit of the sequences \{u_m\}, \{Y_m\},

$$u_m = \sum_{i=1}^m c_i(t)\psi_i(x), \quad Y_m = \sum_{i=1}^m d_i(t)\psi_i(x), \quad m \in \mathbb{N},$$

where $\{\psi_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$.
where \( \{ \psi_i \} \) is the system of eigenfunctions of the problem
\[
(\psi, \phi)_{W_0^{s,2}(\Omega)} = \lambda(\phi, \psi)_{2,\Omega} \quad \forall \phi \in W_0^{s,2}(\Omega)
\]
with a natural \( s \geq 1 + n \max\left\{ 0, \frac{1}{2} - \frac{1}{p} \right\} \), so that \( W_0^{s,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \). The set \( \{ \psi_i \} \) is orthogonal in \( W_0^{s,2}(\Omega) \) and forms an orthonormal basis of \( L^2(\Omega) \). For every finite \( m \) the coefficients \( c_i(t) \) satisfy the system of the nonlinear ordinary differential equations
\[
c_i'(t) = - (|\nabla u_m|^{p-2} \nabla u_m, \nabla \psi_i)_{2,\Omega} + (Y_m + \Theta(x,t,u_m), \psi_i)_{2,\Omega} + (f, \psi_i)_{2,\Omega},
\]
\[
c_i(0) = (u_0, \psi_i)_{2,\Omega},
\]
where \( Y_m \) are solutions of the Volterra equations
\[
Y_m(x,t) = \int_0^t g(t-s)Y_m(x,s)\,ds + \sum_{i=1}^m (F(x,t,u_m), \psi_i)_{2,\Omega} \psi_i.
\]
By Lemma 2.5 for every given \( F \in L^2(\Omega) \) and \( g \in L^2(0,T) \) Eq. (3.9) has a unique solution \( Y_m \in L^2(\Omega) \), by the method of construction \( Y_m \in P_m = \text{span}\{\psi_1, \ldots, \psi_m\} \). Let us denote \( c(t) = (c_1(t), \ldots, c_m(t)) \), \( d(t) = (d_1(t), \ldots, d_m(t)) \), \( f(t) = (f_1(t), \ldots, f_m(t)) \) with \( f_i(t) = (f, \psi_i)_{2,\Omega} \), and set
\[
- (|\nabla u_m|^{p-2} \nabla u_m, \nabla \psi_i)_{2,\Omega} + (Y_m + \Theta(x,t,u_m), \psi_i)_{2,\Omega} = \mathcal{F}_i(t, c(t), d(t)).
\]
System (3.8)–(3.9) can be written in the equivalent form
\[
c_i(t) = c_i(0) + \int_0^t \mathcal{F}_i(s, c(s), d(s))\,ds + \int_0^t f_i(s)\,ds, \quad i = 1, \ldots, m,
\]
\[
d_i(t) = \int_0^t g(t-s)d_i(s)\,ds + (F(x,t,u_m), \psi_i)_{2,\Omega}.
\]
The solutions of system (3.10) define the transformations
\[
c(t) = \mathcal{N}(d(t)), \quad d(t) = \mathcal{M}(c(t))
\]
so that \( c(t) \) is a fixed point of the transformation \( \mathcal{L} = \mathcal{N} \circ \mathcal{M} \). Let us fix some \( \alpha \in (0, 1/2) \) and consider the space of \( m \)-dimensional vectors with Hölder-continuous components
\[
S_{m,\alpha} = \{ v(t) = (v_1(t), \ldots, v_m(t)) : v_i \in C^\alpha[0,T] \}.
\]
Set
\[
\|v(t)\|_{S_{m,\alpha}} = \sum_{i=1}^m \|v_i\|_{C^\alpha[0,T]}, \quad \|v_i\|_{C^\alpha[0,T]} = \sup_{t \in (0,T)} |v_i(t)| + \sup_{t, \tau \in (0,T), t \neq \tau} \frac{|v_i(t) - v_i(\tau)|}{|t - \tau|^\alpha}.
\]
Increasing, if needed, the order \( s \) in the choice of the basis (3.7), for a fixed \( \alpha \in (0, 1/2) \) we estimate
\[
\left( |\nabla u_m|^{p-2} \nabla u_m, \nabla \psi_i \right)_{2,\Omega} \leq C \|\nabla u_m\|_{p,\Omega}^{p-1} \|\nabla \psi_i\|_{p,\Omega}
\]
\[
\leq C \|u_m\|_{H_p^1(\Omega)}^{p-1} \|\psi_i\|_{H_p^1(\Omega)} \leq C \|c(t)\|_{S_{m,\alpha}}^{p-1}
\]
with a constant \( C = C(m, n, p) \) and, by the same token,
\[
\left| (\Theta(x,t,u_m), \psi_i)_{2,\Omega} \right| \leq C \|u_m\|_{\sigma,\Omega}^{\sigma-1} \|\psi_i\|_{\sigma,\Omega} \leq C \|u_m\|_{H_p^1(\Omega)}^{\sigma-1} \|\psi_i\|_{H_p^1(\Omega)}
\]
\[
\leq C(m, n, \sigma) \|c(t)\|_{S_{m,\alpha}}^{\sigma-1},
\]
\[
\left| (F(x,t,u_m), \psi_i)_{2,\Omega} \right| \leq C_1 \left( \|c(t)\|_{S_{m,\alpha}}^{\sigma-1} + \|c(t)\|_{S_{m,\alpha}} + 1 \right)
\]
Proof. Multiplying the equations for Lemma 3.1 energy equality then for every $T < 1$ the operator $L$ maps the closed ball $B = \{c(t) : \|c(t) - c(0)\|_{S_{m, \alpha}} \leq 1\}$ into the set $\|c(t) - c(0)\|_{S_{m, \alpha}} \leq R$, where $R \to 0$ as $T \to 0$. Since the embedding $S_{m, \alpha/2} \subset S_{m, \alpha}$ is compact, and the functions $\Theta(x, t, r)$ and $F(x, t, r)$ are continuous with respect to $r$, it follows from the Schauder fixed point theorem that for a sufficiently small $T = T_m$ the operator $\mathcal{L}$ has at least one fixed point $c^* \in B$. Once $c^*(t)$ is found, the function $Y_m = \sum_{i=1}^{m} d_i(t) \psi_i(x)$ is recovered from the equations for $d_i$ in (3.10).

3.2. A priori estimates

We devote this section to derive independent of $m$ a priori estimates for the solutions of system (3.10). For the sake of simplicity of notation, throughout this subsection we omit the subindex $m$ and denote

$$u := u_m, \quad Y := Y_m, \quad F(x, t, u) := \sum_{i=1}^{m} (F(x, t, u_m), \psi_i)_{2, \Omega} \psi_i.$$

Let us introduce the functional

$$\|u\|_{2(\sigma-1), \Omega} = \|u\|^{2(\sigma-1)}_{2(\sigma-1), \Omega} = \left(\int_{\Omega} |u|^{2(\sigma-1)} \, dx\right)^{1/(2(\sigma-1))}.$$

If $2(\sigma - 1) \geq 1$, this functional becomes the usual norm of the space $L^{2(\sigma-1)}(\Omega)$.

**Lemma 3.1 (Global in Time Estimates).** Let us assume that conditions (3.2)–(3.3) are fulfilled and

either

$$\begin{cases}
1 < \sigma \leq 2, \\
\max \left\{ 1, \frac{2n}{n+2} \right\} < p,
\end{cases} \quad \text{or} \quad \begin{cases}
2 < \sigma \leq 1 + \frac{p}{2}, \\
2 < p.
\end{cases}$$

Then for every $T < \infty$

$$\sup_{(0, T)} \|u(t)\|_{2, \Omega}^2 + \|\nabla u\|_{p, Q(T)}^p \leq C \left(\|f\|_{2, Q}^2 + \|u_0\|_{2, \Omega}^2 + 1\right),$$

$$\|u_t\|_{2, Q}^2 + \|\nabla u\|_{L^{\infty}(0, T; L^p(\Omega))}^p + \sup_{(0, T)} \|u\|_{2(\sigma-1), \Omega}^{2(\sigma-1)} \leq C \left(\|f\|_{2, Q}^2 + \|\nabla u_0\|_{p, \Omega}^p + 1\right).$$

The constant $C$ depends on $n, T, |\Omega|, B, p, \sigma$ but is independent of $m$.

**Proof.** Multiplying the equations for $c_i(t)$ in (3.8) by $c_i(t)$, and summing up the results we arrive at the energy equality

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{2, \Omega}^2) + \|\nabla u(t)\|_{p, \Omega}^p = I_1 + I_2 + I_3,$$

where

$$I_1 = (Y, u)_{2, \Omega}, \quad I_2 = (\Theta(x, t, u), u)_{2, \Omega}, \quad I_3 = (f, u)_{2, \Omega}.$$

Multiplication of equations for $d_i(t)$ by $d_i(t)$ gives

$$\|Y(t)\|_{2, \Omega}^2 = \left(\int_0^t g(t-s)Y(s) \, ds\right)_{2, \Omega} + (F(x, t, u), Y(t))_{2, \Omega}.$$
Following the proof of Lemma 2.4, from this equality we derive the estimate
\[ \|Y(t)\|_{2,\Omega}^2 \leq C (\|F(t)\|_{2,\Omega}^2 + \|F\|_{2,\Omega}^2) \quad \text{for a.e. } t \in (0,T). \]

The second energy equality follows after multiplication of the equations for \( c_i(t) \) by \( c_i'(t) \):
\[ \|u_i(t)\|_{2,\Omega}^2 + \frac{1}{p} \frac{d}{dt} (\|\nabla u_i(t)\|_{p,\Omega}^p) = J_1 + J_2 + J_3 \tag{3.14} \]
with
\[ J_1 = \langle Y, u_i \rangle_{2,\Omega}, \quad J_2 = \langle \Theta(x,t,u), u_i \rangle_{2,\Omega}, \quad J_3 = \langle f, u_i \rangle_{2,\Omega}. \]

Let us denote \( Q_t = \Omega \times (0,t) \). Taking into account (3.3), (2.2) and (2.1), we may estimate
\[ |I_1 + I_2 + I_3| \leq C \left( \|Y\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 + \|u\|_{2\sigma - 2,\Omega}^{2\sigma - 2} + \|f\|_{2,\Omega}^2 \right), \]
\[ |J_1 + J_2 + J_3| \leq \frac{1}{2} \|u(t)\|_{2,\Omega}^2 + C \left( \|Y\|_{2,\Omega}^2 + \|u\|_{2\sigma - 2,\Omega}^{2\sigma - 2} + \|f\|_{2,\Omega}^2 \right), \]
with
\[ \|Y(t)\|_{2,\Omega}^2 \leq C \left( \|F(t)\|_{2,\Omega}^2 + \|F\|_{2,\Omega}^2, Q_t \right) \]
\[ \leq C \left( \|u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2, Q_t + \|u_0\|_{2,\Omega}^2 + \|f\|_{2,\Omega}^2, Q_t + \|u\|_{2\sigma - 2,\Omega}^{2\sigma - 2, Q_t} \right) \]
by virtue of Lemma 2.4. Let us introduce the functions
\[ \Phi(u) := \|u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 + \|u\|_{2\sigma - 2,\Omega}^{2\sigma - 2} + \|u\|_{2\sigma - 2,\Omega}^{2\sigma - 2}, \]
\[ \Psi(f, u_0) := \|f\|_{2,\Omega}^2 + \|f\|_{2,\Omega}^2 + \|u_0\|_{2,\Omega}^2 \tag{3.15} \]
and rewrite (3.13), (3.14) in the form
\[ \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{2,\Omega}^2) + \|\nabla u\|_{p,\Omega}^p \leq C (\Phi(u) + \Psi(f, u_0)), \tag{3.16} \]
\[ \frac{1}{2} \|u(t)\|_{2,\Omega}^2 \left( \frac{d}{dt} (\|u(t)\|_{p,\Omega}^p) \right) \leq C (\Phi(u) + \Psi(f, u_0)). \tag{3.17} \]

The cases \( 1 < \sigma \leq 2 \) and \( \sigma > 2 \) are considered separately.

**Case 1:** \( \sigma \in (1,2] \). In this case \( 0 < 2\sigma - 2 \leq 2 \) and
\[ \Phi(u) \leq C \left\{ \begin{array}{ll} \|u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 & \text{if } \sigma = 2, \\
\|u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 + 1 & \text{if } \sigma < 2. \end{array} \right. \]
It follows from (3.16) that the function \( V(t) = \|u(t)\|_{2,\Omega}^2 \) satisfies the inequality
\[ V(t) \leq V(0) + C \left( \int_0^t V(s) \, ds + \int_0^t \left( \int_0^s V(\xi) \, d\xi + \Psi(f, u_0) + 1 \right) \, ds \right). \]

Integration of this inequality leads to the estimate
\[ V(t) \leq V(0) + C \left( \int_0^t V(s) \, ds + \int_0^t \left( \int_0^s V(\xi) \, d\xi + \Psi(f, u_0) + 1 \right) \, ds \right) \]
\[ \leq V(0) + C(1 + t) \int_0^t V(s) \, ds + C \int_0^t (\Psi(f, u_0) + 1) \, ds \]
whence, by Gronwall’s inequality,
\[ \|u(t)\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \leq C(T)(\|f\|_{2,\Omega}^2 + \|u_0\|_{2,\Omega}^2). \tag{3.18} \]

Estimate (3.11) follows by means of integration of (3.13) in \( t \) and application of (3.18). To obtain (3.12) we integrate (3.14) and make use of (3.18).
Case 2: $\sigma > 2$. It is assumed that $2 < \sigma \leq 1 + \frac{p}{2}$ and $p > 2$, whence $2 < 2\sigma - 2 \leq p$. We will rely on the embedding theorem: $\forall u \in W^{1,p}_0(\Omega)$

$$
\|u\|_{Q, \Omega} \leq C \|\nabla u\|_{p, \Omega}
$$

with $1 \leq q \leq \frac{np}{n-p}$ if $p < n$,

$$
1 \leq q < \infty \quad \text{if } p \geq n.
$$

(3.19)

By Young’s inequality

$$
\|u\|_{2, \Omega}^2 + \|u\|_{2\sigma-2, \Omega}^{2\sigma-2} \leq C \left( \|\nabla u\|_{p, \Omega}^2 + \|\nabla u\|_{p, \Omega}^{2\sigma-2} \right) \leq C \left( \|\nabla u\|_{p, \Omega}^p + 1 \right)
$$

(3.20)

and (3.17) can be written in the form

$$
\frac{1}{2} \|u(t)\|_{2, \Omega}^2 + \frac{1}{p} \frac{d}{dt} \left( \|\nabla u(t)\|_{p, \Omega}^p \right) \leq C \left( \|\nabla u(t)\|_{p, \Omega}^p + \int_0^t \|\nabla u(\tau)\|_{p, \Omega}^p d\tau + \Psi(f, u_0) + 1 \right).
$$

As in the Case 1, by Gronwall’s inequality

$$
\|\nabla u(t)\|_{p, \Omega}^p \leq C(\|\nabla u_0\|_{p, \Omega}^p + \|f\|^2_{2, Q_T} + \|u_0\|^2_{2, \Omega} + 1)
$$

and (3.12) follows. \qed

Lemma 3.2 (Local in Time Estimates). Assume that conditions (3.2)–(3.3) are fulfilled and the exponents $p$, $\sigma$ satisfy one of the following conditions:

(a) $\max \left\{ 1, \frac{2n}{n+2} \right\} < p < n$, $\max \left\{ 2, 1 + \frac{p}{2} \right\} < \sigma \leq 1 + \frac{np}{2n-p}$,

(b) $n \leq p$, $\max \left\{ 2, 1 + \frac{p}{2} \right\} < \sigma < \infty$.

Then there exists $T_{\max} \in (0, T)$ such that for every $t \in (0, T_{\max})$

$$
\|u_t\|_{2, Q_t}^2 + \|\nabla u\|_{L^\infty(0, t; L^p(\Omega))}^p + \|u\|_{L^\infty(0, t; L^{2\sigma-2}(\Omega))}^p \leq C \left( \|f\|^2_{2, Q_T} + \|u_0\|^p_{W^{1,p}_0(\Omega)} + 1 \right)
$$

with an independent of $m$ constant $C = C(n, T, |\Omega|, B, p, \sigma)$.

Proof. If either of assumptions (a), (b) is fulfilled, we use (3.19) and Young’s inequality to find that

$$
\|u\|_{2, \Omega}^2 + \|u\|_{2\sigma-2, \Omega}^{2\sigma-2} \leq C \left( \|\nabla u\|_{p, \Omega}^2 + \|\nabla u\|_{p, \Omega}^{2\sigma-2} \right)
$$

$$
\leq C \left( 1 + \|\nabla u\|_{p, \Omega}^{2\sigma-2} \right) = C \left( 1 + \left( \|\nabla u\|_{p, \Omega}^p \right)^\mu \right) \quad \text{with } \mu = \frac{2(\sigma-1)}{p} > 1.
$$

This inequality entails the estimate on the function $\Phi(u)$:

$$
\Phi(u) \leq C \left( 1 + \left( \|\nabla u\|_{p, \Omega}^p \right)^\mu + \int_0^t \left( \|\nabla u(\tau)\|_{p, \Omega}^p \right)^\mu d\tau \right).
$$

Set $V(t) = \|\nabla u(t)\|_{p, \Omega}^p$. Plugging this estimate on $\Phi(u)$ into (3.17) we arrive at the inequality

$$
V'(t) \leq C \left( 1 + V^\mu(t) + \int_0^t V^\mu(\tau) d\tau \right) + \Psi(f, u_0).
$$

Finally, integrating this inequality over the interval $(0, t)$ we find that

$$
V(t) \leq V(0) + C \left( \int_0^t V^\mu(\tau) d\tau + \int_0^t \int_0^\tau V^\mu(s) d\tau ds \right) + \int_0^T (C + \Psi(f, u_0)) dt
$$

$$
\leq V(0) + C \left( \int_0^t V^\mu(\tau) d\tau + T \int_0^t V^\mu(\tau) d\tau \right) + C_1 \left( 1 + \|f\|^2_{2, Q_T} + \|u_0\|^2_{2, \Omega} \right)
$$

$$
= V(0) + C(1 + T) \int_0^t V^\mu(\tau) d\tau + C_1 \left( 1 + \|f\|^2_{2, Q_T} + \|u_0\|^2_{2, \Omega} \right).$$
Let us write this inequality in the form
\[ V(t) \leq K + C \int_0^t V^\mu(s) \, ds, \]
\[ K = V(0) + C_1 \left( 1 + \|f\|_{L^2(Q_T)}^2 + \|u_0\|_{L^2(\Omega)}^2 \right). \]

It is straightforward to check that the function
\[ W(t) = \frac{K}{(1 - CK^{-1}t)^{\frac{1}{\mu-1}}} \]
solves the equation
\[ W(t) = K + C \int_0^t W^\mu(s) \, ds \quad \text{on the interval } 0 < t < T_{\text{max}} = \frac{1}{CK^{-1}}. \]

Let us check now that \( W(t) \) majorates \( V(t) \) on the interval \( (0, T_{\text{max}}) \). Take an arbitrary \( \epsilon > 0 \) (small) and consider the function
\[ W_\epsilon(t) = \frac{K_\epsilon}{(1 - CK_\epsilon^{-1}t)^{\frac{1}{\mu-1}}}, \quad K_\epsilon = K + \epsilon, \]
which satisfies the equation
\[ W_\epsilon = K_\epsilon + C \int_0^t W_\epsilon^\mu(s) \, ds \quad \text{for } t \in (0, T_\epsilon) \]
with \( T_\epsilon = 1/(CK_\epsilon^{-1}) < T_{\text{max}} \). Let us assume that \( \{t \in [0, T_\epsilon] : V(t) \geq W_\epsilon(t)\} \neq \emptyset \) and set
\[ t_\epsilon = \inf\{t \in (0, T_\epsilon) : V(t) \geq W_\epsilon(t)\}. \]

For every finite \( m \) the functions \( u \in P_m \) and \( V(t) \) are continuous in \([0, T]\), \( W_\epsilon(t) \) is continuous by definition and \( V(0) < W_\epsilon(0) \), which yields the inequality \( t_\epsilon > 0 \). Subtracting the equation for \( W_\epsilon \) from the inequality for \( V \) we find that at the moment \( t = t_\epsilon \)
\[ 0 \leq V(t_\epsilon) - W_\epsilon(t_\epsilon) \leq -\epsilon + C \int_0^{t_\epsilon} (V^\mu(s) - W_\epsilon^\mu(s)) \, ds < -\epsilon \]
because by assumption \( V^\mu(s) < W_\epsilon^\mu(s) \) for \( s \in [0, t_\epsilon] \). This contradiction means that \( V(t) < W_\epsilon(t) \) on \([0, T_\epsilon]\).

Since \( \epsilon > 0 \) is arbitrary, we conclude that \( V(t) \leq W(t) \) for \( t \in [0, T] \). \( \square \)

The derived uniform estimates allow one to continue the sequences \( \{u_m\}, \{Y_m\} \) to an arbitrary time interval \((0, T)\) under the conditions of Lemma 3.1, or to the maximal interval \((0, T_{\text{max}})\) under the conditions of Lemma 3.2.

### 3.3. Existence of weak solutions

**Theorem 3.1.** Let us assume that
\[ f \in L^2(Q_T), \quad u_0 \in W_0^{1,p}(\Omega), \quad g, g' \in L^2(0, T), \quad |g(0)| < \infty, \quad p > \max\left\{1, \frac{2n}{2+n} \right\}. \]

If \( \Theta \) satisfies the growth condition (3.3), then problem (3.1) has at least one weak solution in the sense of Definition 1.1. Under the conditions of Lemma 3.1 this is a global solution which exists on every time interval \([0, T]\), under the conditions of Lemma 3.2 the solution exists on every time interval \((0, T)\) with \( T < T_{\text{max}} \).
Proof. By Lemmas 3.1, 3.2

1. \( u_m \) are uniformly bounded in \( L^\infty(0,T;W_0^1,\Omega) \cap L^\infty(0,T;L^2(\Omega)) \),
2. \( (u_m)_t \) are uniformly bounded in \( L^2(0,T;L^2(\Omega)) \).

It follows from [25, Sec.9, Cor.6] that the sequence \( \{u_m\} \) is relatively compact in \( L^q(0,T;L^2(\Omega)) \) with any \( 1 < q < \infty \). Thus, there exist \( u \in L^2(Q_T), Y \in L^2(Q_T) \) and \( A \in (L^p(Q_T))^n \) such that

1. \( u_m \to u \) a.e. in \( Q_T \) and \( L^2(Q_T) \),
2. \( \Theta(x,t,u_m) \to \Theta(x,t,u) \) a.e. in \( Q_T \) and weakly in \( L^r(Q_T) \),
3. \( u_m \to u \) in \( L^q(Q_T) \),
4. \( u_{mt} \to u_t \) in \( L^2(Q_T) \),
5. \( |\nabla u_m|^{p-2} \nabla u_m \to A \) in \( (L^p(Q_T))^n \),
6. \( Y_m \to Y \) in \( L^2(Q_T) \).

By the method of construction of \( u_m \), for every \( \phi \in \mathcal{P}_N = \text{span}\{\psi_1, \ldots, \psi_N\} \) with \( N \leq m \)

\[
\int_{Q_T} (u_{mt}\phi + \nabla \phi \cdot |\nabla u_m|^{p-2} \nabla u_m - (Y_m + \Theta(x,\tau,u_m) + f)\phi)\,dx\,d\tau = 0 \tag{3.22}
\]
and, in particular,

\[
\int_{Q_T} |u_{mt}u_m + |\nabla u_m|^{p-2} \nabla u_m - (Y_m + \Theta(x,\tau,u_m) + f)u_m|\,dx\,d\tau = 0. \tag{3.23}
\]

Letting in (3.22) \( m \to \infty \), for every \( \phi \in \mathcal{P}_N \)

\[
\int_{Q_T} [u_t\phi + \nabla \phi \cdot A - (Y + \Theta(x,\tau,u) + f)\phi]\,dx\,d\tau = 0. \tag{3.24}
\]

Since the set \( \{\phi_N\}_{N \geq 1} \) is dense in \( W(Q) \), the previous equality is true for every \( \phi \in W(Q_T) \) and, in particular, for \( \phi = u \):

\[
\int_{Q_T} [u_tu + \nabla u \cdot A - (Y + \Theta(x,\tau,u) + f)u]\,dx\,d\tau = 0. \tag{3.25}
\]

Now we need to prove that for every admissible test-function \( \phi \)

\[
\int_{Q_T} \nabla \phi \cdot |\nabla u_m|^{p-2} \nabla u_m \,dt\,dx \to \int_{Q_T} \nabla \phi \cdot |\nabla u|^{p-2} \nabla u \,dt\,dx.
\]

By monotonicity, for every smooth function \( \zeta \)

\[
|\nabla u_m|^p = (|\nabla u_m|^{p-2} \nabla u_m - |\nabla \zeta|^{p-2} \nabla \zeta) \cdot \nabla (u_m - \zeta) + |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \zeta \geq |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \zeta \tag{3.26}
\]

Subtracting (3.23) from (3.25) we obtain the equality

\[
\int_{Q_T} (-\nabla u \cdot A + |\nabla u_m|^p) \,dx\,d\tau = \sum_{i=1}^{4} I_{i,m} \tag{3.27}
\]

with

\[
I_{1,m} = -\int_{Q_T} (u_{mt}u_m - u_t u) \,dx\,d\tau, \quad I_{2,m} = -\int_{Q_T} (Y u - Y_m u_m) \,dx\,d\tau,
\]

\[
I_{3,m} = -\int_{Q_T} f(u - u_m) \,dx\,d\tau, \quad I_{4,m} = -\int_{Q_T} (\Theta(x,\tau,u) - \Theta(x,\tau,u_m)) u_m \,dx\,d\tau.
\]
The integrals $I_{1,m}$, $I_{2,m}$, $I_{3,m}$ tend to zero as $m \to \infty$ because $u_m u_{mt}$, $Y_m u_m$, $fu_m$ are the products of weakly and strongly converging sequences. Since $\Theta$ is a Carathéodory function, $I_{4,m} \to 0$ because of the weak convergence (3.21). and the pointwise convergence $u_m - u \to 0$ in $Q_T$. Due to the monotonicity condition (3.26), equality (3.27) yields the inequality
\[ \int_{Q_T} \left( |\nabla \zeta|^p - 2 \nabla \zeta \cdot \nabla (u_m - \zeta) + |\nabla u_m|^p - 2 \nabla u_m \cdot \nabla \zeta - \nabla u \cdot A \right) \, dx \, dt \leq \sum_{i=1}^{4} I_{i,m}. \]
Letting $m \to \infty$ we obtain the inequality
\[ \int_{Q_T} (|\nabla \zeta|^p - 2 \nabla \zeta - A) \cdot \nabla (u - \zeta) \, dx \, dt \leq 0 \]
with an arbitrary test-function $\zeta \in W(Q_T)$. Let $\zeta = u \pm \delta \eta$ with a positive parameter $\delta > 0$ and an arbitrary $\eta \in W$. Simplifying the resulting inequality and letting $\delta \to 0$ we arrive at the inequalities
\[ \pm \int_{Q_T} (|\nabla u|^p - 2 \nabla u - A) \cdot \nabla \eta \, dx \, dt \geq 0 \quad \forall \eta \in W(Q_T), \]
which is impossible unless $A = |\nabla u|^p - 2 \nabla u$ a.e. in $Q_T$. Reverting to (3.24) we conclude that
\[ \int_{Q_T} \left[ u_t \phi + \nabla \phi \cdot |\nabla u|^p - 2 \nabla u - (Y + \Theta(x, \tau, u) + f) \phi \right] \, dx \, dt = 0. \quad (3.28) \]
It follows that $\Delta_p u = -u_t + Y + f + \Theta(x, t, u) \in L^2(Q_T)$ and Eq. (3.4) is fulfilled a.e. in $Q_T$. Moreover, the inclusions $u, u_t \in L^2(Q_T)$ yield the inclusion $u \in C([0, T]; L^2(\Omega))$.

Applying the derived convergence properties of the sequence $\{u_m\}$ and (3.21) it is easy to see that
\[ \forall \chi \in W_0^{1,p}(\Omega) \quad \int_{\Omega} F(x, t, u_m) \chi(x) \, dx \to \int_{\Omega} F(x, t, u) \chi(x) \, dx \quad \text{as } m \to \infty \]
and that the limit function $Y$ satisfies the identity
\[ \int_{\Omega} Y(x, t) \chi(x) \, dx = \int_{Q_T} \chi(x) g(t - s) Y(x, s) \, ds \, dx + \int_{\Omega} F(x, t, u) \chi(x) \, dx \quad \forall t \in (0, T). \quad (3.29) \]
To identify $Y$ we test (3.28) in the cylinder $Q_t$ with $\phi(x, s) = \chi(x) g(t - s)$, $\chi(x) \in C_0^\infty(\Omega)$, and compare the result with (3.29): for a.e. $t \in (0, T)$
\begin{align*}
\int_{\Omega} \phi(x) \int_0^t g(t - s) \Delta_p u(s) \, ds \, dx &= \int_{\Omega} \phi(x) \int_0^t g(t - s) Y(s) \, ds \, dx + \int_{\Omega} \phi(x) \int_0^t g(t - s) (\Theta(x, t, u) + f(x, s)) \, ds \, dx \\
&\quad - \int_{\Omega} \phi(x) \int_0^t g(t - s) u_t(x, s) \, ds \, dx \\
&= \int_{\Omega} \phi(x) \int_0^t g(t - s) Y(s) \, ds \, dx + \int_{\Omega} \phi(x) F(x, t, u) \, dx = \int_{\Omega} \phi(x) Y(x, t) \, dx.
\end{align*}
Since $t \in (0, T)$ is arbitrary, it follows that
\[ Y(x, t) = \int_0^t g(t - s) \Delta_p u(x, s) \, ds \quad \text{a.e. in } Q_T. \]

4. Uniqueness of weak solutions

Let us first check that every weak solution of problem (3.1) generates a solution of system (3.4)–(3.5) with $F$ defined by (3.6). Let $u$ be a weak solution of problem (3.1). Since the equation is fulfilled a.e. in $Q,$
for every \( \psi(x) \in C_0^\infty(\Omega) \) and a.e. \( t \in (0, T) \) multiplication of Eq. (3.4) by \( \psi(x)g(t - \tau) \) and integration over the cylinder \( \Omega \times (0, t) \) give

\[
\int_{\Omega} \psi(x) \left( \int_0^t g(t - \tau) \Delta_p u(\tau) \, d\tau - \int_0^t g(t - \tau) \left( \int_0^\tau g(\tau - s) \Delta_p u(s) \, ds \right) \, d\tau \right) \, dx
= \int_{\Omega} \psi(x) (u(t)g(0) - u_0g(t)) \, dx \\
- \int_{\Omega} \psi(x) \int_0^t (g(t - \tau) \Theta(x, \tau, u(\tau)) - g'(t - \tau)u(\tau) + g(t - \tau)f(x, \tau)) \, dx \, d\tau
= \int_{\Omega} \psi(x) F(x, t, u(\cdot)) \, dx.
\]

It follows that the function \( Y = \int_0^t g(t - s) \Delta_p u(s) \, ds \) is a solution of the Volterra equation (3.5).

**Theorem 4.1.** Let us assume that

\[
1 < p < \infty, \quad g, g' \in L^2(0, T), \quad |g(0)| < \infty, \\
|\Theta(x, t, u_1) - \Theta(x, t, u_2)| \leq L|u_1 - u_2|, \quad L = \text{const} > 0.
\]

Then problem (3.1) has at most one weak solution.

**Proof.** Let \( u_1, u_2 \) be two different solutions of problem (3.1) and \((u_1, Y_1), (u_2, Y_2)\) be the corresponding solutions of system (3.4)–(3.5). Set \( u = u_1 - u_2 \) and \( Y = Y_1 - Y_2 \). Subtracting relations (3.28) for \( u_1 \) with the test-function \( \phi = u \) we find that

\[
\frac{1}{2} \|u(t)\|_{2, \Omega}^2 + \int_0^t \int_{\Omega} \nabla u \cdot (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \, dx \, d\tau = I_1 + I_2,
\]

where

\[
I_1(t) = \int_0^t \int_{\Omega} Y u \, dx \, d\tau, \quad I_2(t) = \int_0^t \int_{\Omega} (\Theta(x, \tau, u_1) - \Theta(x, \tau, u_2))u \, dx \, d\tau.
\]

On the other hand,

\[
Y(x, t) = \int_0^t g(t - s)Y(x, s)ds + \tilde{F}(x, t) \quad \text{a.e. in } Q
\]

with

\[
\tilde{F}(x, t) = u(x, t)g(0) - \int_0^t g'(t - s)u(x, s) \, ds + \int_0^t g(t - s)(\Theta(x, s, u_1) - \Theta(x, s, u_2)) \, ds.
\]

There is a constant \( C = C(L, |g(0)|, \|g\|_{2,(0,T)}, \|g'\|_{2,(0,T)}) \) such that

\[
\|\tilde{F}(t)\|_{2, \Omega}^2 \leq C \left[ \|u(t)\|_{2, \Omega}^2 + \int_0^t \|u(s)\|_{2, \Omega}^2 \, ds \right],
\]

\[
\|\tilde{F}\|_{2, Q}^2 = \int_0^t \|\tilde{F}(s)\|_{2, \Omega}^2 \, ds \leq C(1 + T) \int_0^t \|u(s)\|_{2, \Omega}^2 \, ds
\]

whence, by virtue of Lemma 2.4

\[
\|Y(t)\|_{2, \Omega}^2 \leq C \|\tilde{F}(t)\|_{2, \Omega}^2 \leq C \left[ \|u(t)\|_{2, \Omega}^2 + \int_0^t \|u(s)\|_{2, \Omega}^2 \, ds \right].
\]
Applying the last estimate and Young’s inequality we obtain
\[ |I_1(t)| \leq \int_0^t \|u(s)\|_{2,\Omega} \|Y(s)\|_{2,\Omega} ds \leq C \int_0^t \|u(s)\|_{2,\Omega}^2 + \int_0^s \|u(\tau)\|_{2,\Omega}^2 d\tau \frac{ds}{\tau} \leq C \int_0^t \left( \|u(s)\|_{2,\Omega}^2 + \int_0^s \|u(\tau)\|_{2,\Omega}^2 d\tau \right) ds, \]
\[ |I_2(t)| \leq L \int_0^t \int_\Omega u^2(x,s) \, dxds = L \int_0^t \|u(s)\|_{2,\Omega}^2 ds, \]
with constants $C$ depending on $L_1, \|g(0)\|, \|g\|_{2,(0,T)}$ and $\|\dot{g}\|_{2,(0,T)}$. Plugging these inequalities into (4.1) and using the monotonicity of the second term on the left-hand side we arrive at the inequality
\[ \|u(t)\|_{2,\Omega}^2 \leq C \int_0^t \left( \|u(s)\|_{2,\Omega}^2 + \int_0^s \|u(\tau)\|_{2,\Omega}^2 d\tau \right) ds \leq C(1 + t) \int_0^t \|u(s)\|_{2,\Omega}^2 ds. \]
By Gronwall’s inequality $\|u(t)\|_{2,\Omega}^2 = 0$ for all $t \in (0,T)$. \hfill \Box

5. Finite speed of propagation and waiting time

Let us fix a point $x_0 \in \Omega$, a number $0 < \rho_0 < \text{dist}(x_0, \partial\Omega)$ and accept the notation
\[ B_\rho(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < \rho \}, \quad S_\rho = \partial B_\rho(x_0), \]
\[ Q_{\rho,t} = B_\rho(x_0) \times (0,t), \quad S_{\rho,t} = S_\rho(x_0) \times (0,t). \]
We are interested in the property of finite speed of propagation from the initial data. This property is local and depends only on the nonlinear structure of the equation, for this reason we study the local weak solutions $u \in W$ of Eq. (1.1). Let us denote
\[ W(Q_{\rho_0,T}) = L^2(Q_{\rho_0,T}) \cap L^p(0,T; W^{1,p}(B_{\rho_0}(x_0))). \]
We say that a function $u \in C([0,T]; L^2(B_{\rho_0})) \cap W(Q_{\rho_0,T})$ with $u_t \in L^2(Q_{\rho_0,T})$ is a local weak solution of Eq. (1.1) if for every test-function $\phi \in W(Q_{\rho_0,T})$ such that $\phi = 0$ on $S_{\rho_0} \times (0,T)$
\[ \int_{Q_{\rho_0,T}} \left( u_t \phi + \nabla \phi \cdot (|\nabla u|^{p-2} \nabla u + \int_0^t g(t-s) |\nabla u(s)|^{p-2} \nabla u(s) \, ds) \right) \, dxdt = \int_{Q_{\rho_0,T}} (f + \Theta(x,t,u)) \phi \, dxdt. \quad (5.1) \]
It is clear that every weak solution of problem (1.1) constructed in Theorem 3.1 is a local weak solution.

For every $u \in C([0,T]; L^2(B_{\rho_0})) \cap W(Q_{\rho_0,T})$, we introduce the energy functions
\[ E(\rho,t) = \int_0^t \int_{B_\rho} |\nabla u(x,\tau)|^p \, dx \, d\tau \]
\[ b(\rho,t) = \int_{B_\rho} u^2(x,t) \, dx, \quad \overline{b}(\rho,t) = \text{ess sup}_{\tau \in (0,t)} b(\rho,\tau). \]
By the Lebesgue differentiation theorem for a.e. \( \rho \in (0, \rho_0) \) there exists
\[
E_\rho(\rho, t) = \int_0^t \int_{S_\rho} |\nabla u|^p \, dS \, dt \quad \text{for a.e.} \quad \rho \in (0, \rho_0).
\]

We will consider the local weak solutions with finite energy:
\[
\mathcal{B}(\rho, T) + E(\rho_0, T) = \text{ess sup}_{(0, T)} \|u(\cdot, t)\|^2_{L^2(B_{\rho_0})} + \int_{Q_{\rho_0, T}} |\nabla u|^p \, dx \, dt \leq L \quad (5.2)
\]
with a finite constant \( L \). This assumption is surely fulfilled for the weak solutions of problem (1.1).

### 5.1. The energy relations

**Lemma 5.1.** Let \( B_{\rho_0}(x_0) \subset \Omega \). Assume that
\[
u_0(x) = 0 \quad \text{in} \quad B_{\rho_0}, \quad f \equiv 0 \quad \text{in} \quad Q_{\rho_0}, \quad g \in L^p(0, T).
\]

Then for every local weak solution of Eq. (1.1) satisfying condition (5.2), the energy equality holds:
\[
\forall \text{a.e.} \rho \in (0, \rho_0), \quad \forall t \in (0, T)
\]
\[
\frac{1}{2} b(\rho, t) + E(\rho, t) - \int_0^t \int_{B_\rho} u \Theta(x, t, u) \, dx \, dt = I_1 + I_2 + I_3, \quad (5.3)
\]
where
\[
I_1 = \int_0^t \int_{\partial S_\rho} u(\tau) |\nabla u(\tau)|^{p-2} \nabla u(\tau) \cdot \nu \, dS \, d\tau,
I_2 = - \int_0^t \int_{B_\rho} \nabla u(\tau) \int_0^{\tau} g(\tau - s) |\nabla u(s)|^{p-2} \nabla u(s) \, ds \, dx \, d\tau,
I_3 = \int_0^t \int_{\partial S_\rho} u(\tau) \int_0^{\tau} g(\tau - s) |\nabla u(s)|^{p-2} \nabla u(s) \cdot \nu \, ds \, dS \, d\tau,
\]
and \( \nu \) denotes the unit outer normal to \( S_\rho \).

**Proof.** Let us denote \( r = |x - x_0| \) and introduce the functions
\[
\zeta_k(r) = \begin{cases} 
0 & \text{if } r \geq \rho, \\
(k(\rho - r)) & \text{if } r \in [\rho - 1/k, \rho], \quad k \in \mathbb{N}, \\
1 & \text{if } r < \rho - 1/k,
\end{cases}
\]
Choosing \( \phi = u \zeta_k(r) \) for the test-function in (5.1) we obtain the equality
\[
\frac{1}{2} \int_{Q_{\rho_0, T}} (u^2(t)) \zeta_k(r) \, dx \, dt + \int_{Q_{\rho_0, T}} \zeta_k(r) \left( |\nabla u(t)|^p + \nabla u(t) \int_0^t g(t - s) |\nabla u(s)|^{p-2} \nabla u(s) \, ds \right) \, dx \, dt
\]
\[
= k \int_0^T \int_{\rho < |x - x_0| < \rho + \frac{1}{k}} u(t) \left( |\nabla u(t)|^{p-2} \nabla u(t) + \int_0^t g(t - s) |\nabla u(s)|^{p-2} \nabla u(s) \, ds \right) \cdot \nu \, dx \, dt
\]
\[
+ \int_{Q_{\rho_0, T}} \Theta(x, t, u(t)) u(t) \zeta_k(r) \, dx \, dt. \quad (5.4)
\]

By the dominated convergence theorem every term on the left-hand side and the last term of the right-hand side has a limit as \( k \to \infty \). Let us denote
\[
J_1(x) = \int_0^T |u(t)| |\nabla u(t)|^{p-2} \nabla u(t) \, dt, \quad J_2(x) = \int_0^T |u(t)| \left( \int_0^t |g(t - s)| |\nabla u(s)|^{p-1} \, ds \right) \, dt.
\]
Since for every local weak solution
\[ \int_{B_p} |J_1(x)| \, dx \leq \|u\|_{p,Q_p,T} \|\nabla u\|_{p,Q_p,T}^{\frac{p}{p-1}}, \]
\[ \int_{B_p} |J_2(x)| \, dx \leq \|u\|_{p,Q_p,T} \left( \int_{Q_p,T} \left( \int_0^t |g(t-s)| \|\nabla u(s)\|^{p-1} \, ds \right)^{\frac{1}{p-1}} \right)^{\frac{p}{p'}}, \]
it follows from the Lebesgue differentiation theorem that for a.e. \( \rho \in (0, \rho_0) \) there exists
\[ \lim_{k \to \infty} k \int_0^T \int_{\rho < \rho \leq \rho + \frac{1}{k}} u(t) \left( |\nabla u(s)|^{p-2} \nabla u(s) + \int_0^t g(t-s) |\nabla u(s)|^{p-2} \nabla u(s) \, ds \right) \cdot \nu \, dx \, dt = I_1 + I_3. \]

Letting in (5.4) \( k \to \infty \) we obtain (5.3) with \( t = T \). The arguments remain valid if we substitute \( T \) by any \( t \in (0, T) \). \( \square \)

**Lemma 5.2.** Let the conditions of Lemma 5.1 be fulfilled and
\[ s \Theta(x,t,s) \leq 0 \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \text{a.e.} \quad (x,t) \in Q_{\rho_0,T}. \tag{5.5} \]
Then the energy \( E(\rho, t) \) satisfies the differential inequality: for a.e. \( \rho \in (0, \rho_0) \)
\[ \frac{1}{2} b(\rho, t) + E(\rho, t) \leq \left( 1 + t^{\frac{1}{p'}} \|g\|_{p,(0,T)} \right) \|u\|_{p,S_p,t} \left( E(\rho, t) \right)^{\frac{1}{p'}} + t^{\frac{1}{p'}} \|g\|_{p} E(\rho, t), \tag{5.6} \]
where \( E \) and \( b \) are considered as functions of \( \rho \) and depend on \( t \) as a parameter.

**Proof.** To obtain (5.6) it is sufficient to estimate the terms on the right-hand of (5.3):
\[ |I_1| \leq \|u\|_{p,S_p,t} |\nabla u|^{p-1}_{p,S_p,t} = \|u\|_{p,S_p,t} \left( E(\rho, t) \right)^{\frac{1}{p'}}, \]
\[ |I_2| \leq \|g\|_{p,(0,T)} \int_{Q_{p,t}} |\nabla u|^{\frac{1}{p}} \left( \int_0^T |\nabla u(s)|^{p} \, ds \right)^{\frac{1}{p'}} \, dx \, dt \]
\[ \leq \|g\|_{p,(0,T)} \|\nabla u\|_{p,Q_p,t} \left( \int_{Q_{p,t}} \int_0^T |\nabla u(s)|^{p} \, ds \, dx \, dt \right)^{\frac{1}{p'}} \]
\[ \leq t^{\frac{1}{p'}} \|g\|_{p,(0,T)} \|\nabla u\|_{p,Q_p,t} = t^{\frac{1}{p'}} \|g\|_{p,(0,T)} E(\rho, t), \]
\[ |I_3| \leq \|g\|_{p,(0,T)} \int_{S_p,t} |u| \left( \int_0^T |\nabla u(s)|^{p} \, ds \right)^{\frac{1}{p'}} \, dS \, d\tau \]
\[ \leq \|g\|_{p,(0,T)} \|u\|_{p,S_p,t} \left( \int_{S_p,t} \left( \int_0^T |\nabla u(s)|^{p} \, ds \right) \, dS \right)^{\frac{1}{p'}} \]
\[ \leq t^{\frac{1}{p'}} \|g\|_{p,(0,T)} \|u\|_{p,S_p,t} \|\nabla u\|_{p,S_p,t} = t^{\frac{1}{p'}} \|g\|_{p,(0,T)} \left( E(\rho, t) \right)^{\frac{1}{p'}}. \quad \square \]

### 5.2. Finite speed of propagation

**Theorem 5.1 (Finite Speed of Propagation).** Let \( p > 2 \) and \( g \in L^p(0,T) \). Let
\[ t_* = \min \left\{ 1, \frac{1}{\left( 2\|g\|_{p,(0,T)} \right)^{\frac{1}{p'}}} \right\}, \quad \rho_0^{-p\delta} \leq t_*, \quad \delta = - \left( 1 + \frac{n(p-2)}{2p} \right). \]
Assume that \( u_0 = 0 \) in \( B_{\rho_0} \) and \( f = 0 \) in \( Q_{\rho_0,T} \). If \( \Theta \) satisfies condition (5.5), then every local weak solution of Eq. (1.1) in \( Q_{\rho_0,T} \), satisfying (5.2), possesses the property of finite speed of propagation:

\[
u(x,t) = 0 \quad \text{a.e. in } B_{\rho(t)}(x_0), \quad t \in (0,t_*),
\]

where \( \rho(t) \) is given by the formula

\[
\rho^\mu(t) = \max \left\{ 0, \rho_0^\mu - \frac{M \mu t_{\rho_0}^{p-1}}{1 - \gamma} L^{1-\gamma} t^{\frac{1-\gamma}{p-1}} \right\}
\]

with the constants \( M, \gamma, \mu, \delta \) defined in formulas (5.14), (5.16).

**Proof.** Using in (5.6) the inequality \( t^{\frac{p-1}{p}} \|g\|_{p,(0,T)} < \frac{1}{2} \) we transform it into the form

\[
b(\rho, t) + E(\rho, t) \leq 3\|u\|_{p,S_\rho,t} E^{\frac{1}{p}}_\rho (\rho, t) \quad \text{for a.e. } \rho \in (0, \rho_0).
\]

Let us make use of the trace-interpolation inequality: \( \forall u \in W^{1,p}(B_{\rho_0}) \cap L^2(B_{\rho_0}) \)

\[
\|u\|_{p,S_\rho} \leq C(p,n) \left( \|\nabla u\|_{p,B_\rho} + \rho^\delta \|u\|_{2,B_\rho} \right) \|u\|_{1,\rho}^{1-\theta}
\]

with the exponents

\[
\theta = \frac{n(p - 2) + 2}{n(p - 2) + 2p} < 1, \quad \delta = - \left( 1 + \frac{n(p - 2)}{2p} \right) < -1.
\]

The detailed proof of inequality (5.8) can be found in [11, Sec.4]. Rising both sides of (5.8) to the power \( p \) and integrating in \( t \) we get

\[
\|u\|_{p,S_\rho,t} \leq C t^{\frac{1-\theta}{p}} b^{\frac{1-\theta}{p}} (\rho, t) \left( E^{\frac{1-p}{p}}_\rho (\rho, t) + \rho^\delta t^{\frac{1}{p}} b^{\frac{1}{p}} (\rho, t) \right)^\theta,
\]

whence by virtue of (5.7)

\[
\bar{b} + E \leq C E^{\frac{p-1}{p}}_\rho t^{\frac{1-\theta}{p}} b^{\frac{1-\theta}{p}} \left( E^{\frac{1-p}{p}}_\rho + \rho^\delta t^{\frac{1}{p}} b^{\frac{1}{p}} \right)^\theta
\]

with a constant \( C = C(p,n) \). By assumption \( \bar{b} \leq L \) and \( p > 2 \), which allows one to estimate

\[
\bar{b}^\frac{1}{\theta} = b^{\frac{1}{\theta} - \frac{1}{p}} b^{\frac{1}{\theta}} \leq L^{\frac{1}{\theta}} (\bar{b} + E)^{\frac{1}{\theta}}, \quad L_1 = \max\{1, L\}.
\]

Using this inequality in (5.11) we rewrite the latter in the form

\[
\bar{b} + E \leq C L^{\frac{2}{\gamma} - \frac{1}{\theta}}_1 t^{\frac{1-\theta}{p}} E^{\frac{p-1}{p}}_\rho (\bar{b} + E)^{\theta + \frac{1-\theta}{p}} \max \left\{ 1, t^{\frac{1}{p}} \rho^\delta \right\},
\]

which yields the differential inequality for the energy function \( E \): for a.e. \( \rho \in (0, \rho_0) \) and every \( t \in (0, t_*) \)

\[
E^{\gamma} \leq (\bar{b} + E)^{\gamma} \leq M t^{\frac{1-\theta}{p}} \rho^{\frac{p-1}{p}} \max \left\{ 1, t^{\frac{1}{p}} \rho^\delta \right\} E_\rho
\]

with the constants

\[
M = \left( C L^{\frac{2}{\gamma} - \frac{1}{\theta}}_1 \right)^{-\frac{p}{p-1}}, \quad \gamma = \left( 1 - \frac{\theta}{p} - \frac{1-\theta}{2} \right) \frac{p}{p-1} \in (0, 1),
\]
provided that $p > 2$. Let us claim that $\rho_0^\mu t_* \geq 1$. It follows that max $\left\{ 1, t^\frac{1}{\rho - 1} \rho_0^{\frac{\mu}{\rho - 1}} \right\} \leq t^\frac{1}{\rho - 1} \rho_0^{\frac{\mu}{\rho - 1}}$ for $\rho \in (0, \rho_0)$ and inequality (5.13) can be continued as follows:

$$E^\gamma(\rho, t) \leq Mt^\frac{1-\theta}{\rho - 1} t^\frac{1}{\rho - 1} \rho_0^{\frac{\mu}{\rho - 1}} E(\rho, t) \quad \text{a.e. in } (0, \rho_0), \ 0 \leq E(\rho, t) \leq L.$$  

(5.15)

Let us fix some $t \in (0, t_*)$. If $E(\rho_0, t) = 0$, we are done: since $E$ is nonnegative, monotone increasing and continuous in each of the variables $\rho$, $t$, it is necessary that $E = 0$ and $u = \text{const}$ in $Q_{\rho_0, t}$, whence $u = 0$ because $u(x, 0) = 0$ in $B_{\rho_0}$ by assumption. Let us assume that $E(\rho_0, t) > 0$. By continuity of $E(\rho, t)$ with respect to $\rho$ there is an interval $(\rho, \rho_0)$ where $E > 0$. Integrating (5.15) in the limits $(\rho, \rho_0)$ we obtain the inequality

$$0 \leq E^{1-\gamma}(\rho, t) \leq E^{1-\gamma}(\rho_0, t) - \frac{1-\gamma}{M} t^\frac{1-\theta}{\rho - 1} t_*^\frac{\mu}{\rho - 1} \frac{1}{\mu} (\rho_0^\mu - \rho^\mu), \quad \mu = 1 - \frac{p\delta}{p-1} > 1.$$  

(5.16)

Since $E(\rho_0, t) \leq L$, it follows from (5.16) that for the fixed $t \in (0, t_*)$ the energy function $E(\rho, t)$ must be zero whereas

$$\rho^\mu \leq \rho_0^\mu - \frac{M\mu t_*^{\frac{1}{\rho - 1}}}{1-\gamma} L^{1-\gamma} t^\frac{1-\theta}{\rho - 1}. \quad \Box$$

5.3. The waiting time property

Let us consider the following situation: there exist $R > 0$ and $\rho_0 \in (0, R)$ such that for some point $x_0 \in \Omega$

$$B_R \subseteq \Omega, \quad u_0 \equiv 0 \text{ in } B_{\rho_0}, \quad f \equiv 0 \text{ in } Q_{\rho_0, T}.$$  

(5.17)

The supports of $u_0$, $f$ are contained in $\Omega \setminus B_{\rho_0}$ and $Q \setminus Q_{\rho_0, T}$. We will assume that $u_0$ and $f$ are sufficiently “flat” near the boundaries of their supports: there exist $\epsilon > 0$ such that

$$F(\rho, t) = \|u_0\|_{2, B_{\rho}}^2 + \|f\|_{2, Q_{\rho, t}}^2 \leq \epsilon(\rho - \rho_0)_{+}^{\frac{1}{\rho - 1}} \forall \rho \in (\rho_0, R), \ t \in (0, T)$$  

(5.18)

with the exponents

$$\nu = \frac{p}{2(p-1)} \left( 1 + \frac{p-2}{p} \theta \right) < 1$$

and $\theta$ from condition (5.9).

**Theorem 5.2.** Let $u$ be a local weak solution of Eq. (1.1) with $p > 2$ and $\Theta(x, t, u)$ satisfying condition (5.5). Let condition (5.2) be fulfilled with a constant $L$. Assume that the data $u_0$, $f$ satisfy (5.17) and (5.18) with a constant $\epsilon$. If $\epsilon$ is sufficiently small, there exists $t_* \in (0, T)$, which depends on $L$, $\|g\|_{p,(0,T)}$ and $\epsilon$, such that

$$u(x, t) = 0 \quad \text{a.e. in } B_{\rho_0} \times (0, t_*).$$

**Proof.** Following the proof of Lemma 5.1 we derive the equality: $\forall \text{a.e. } \rho \in (\rho_0, R), \forall t \in (0, T)$

$$\frac{1}{2} b(\rho, t) + E(\rho, t) - \int_0^t \int_{B_{\rho}} u \Theta(x, t, u) dx dt = I_1 + I_2 + I_3 + I_4,$$

with $I_1$, $I_2$, $I_3$ defined in Lemma 5.1 and

$$I_4 = \frac{1}{2} \|u_0\|_{2, B_{\rho}}^2 + \int_0^t \int_{B_{\rho}} fu dx dt.$$
By the Cauchy inequality and due to assumption (5.18)

$$|I_4| \leq \frac{1}{2} \bar{b}(\rho, t) + \frac{1}{2} \left( \|u_0\|_{L^2_{\rho, t}^p} + \int_0^t \int_{B_{\rho}} f^2 \, dx \, dt \right) \leq \frac{\epsilon}{2} (\rho - \rho_0)^{\frac{1-\nu}{\nu}} + \frac{1}{2} \bar{b}(\rho, t).$$

Assuming $t^{\frac{1}{\nu}} < \min \left\{ \frac{1}{2|\rho|_{L^p((0,T)}, 1 \right\}$ and repeating the proof of Lemma 5.2 we obtain the nonhomogeneous analogue of inequality (5.7),

$$b(\rho, t) + E(\rho, t) \leq 3\|u\|_{L^p_{\rho, t}} (E_\rho(\rho, t))^{\frac{\nu}{\nu-1}} + 2\epsilon(\rho - \rho_0)^{\frac{1-\nu}{\nu}}, \quad \rho \in (\rho_0, R),$$

which can be transformed with the help of (5.10) and (5.12):

$$\bar{b} + E(\rho, t) \leq K t^{\frac{1-\theta}{\theta}} (E + \bar{b})^{\frac{\theta}{\theta-1}} (E_\rho(\rho, t))^{\frac{\nu}{\nu-1}} + 2\epsilon(\rho - \rho_0)^{\frac{1-\nu}{\nu}}, \quad \rho \in (\rho_0, R),$$

where

$$K = C \max \left\{ 1, \rho_0^{\theta \delta} \right\} \max \left\{ 1, T^{\frac{\theta}{2}} \right\}$$

and $C$ is an absolute constant. Applying Young’s inequality to the first term on the right-hand side, simplifying and raising both parts to the power $\nu$ we arrive at the inequality

$$E' \leq (\bar{b} + E)^\nu \leq C \left( \left( K \nu t^{\frac{1-\theta}{\theta}} \right)^{\frac{\lambda}{\lambda - \nu}} E_\rho + \epsilon^\nu (\rho - \rho_0)^{\nu/(1-\nu)} \right)$$

with

$$\frac{1}{\lambda} = \frac{\theta}{\nu} + \frac{1 - \theta}{2} < 1.$$

Let us fix some $t_* > 0$, $\epsilon_* > 0$ and consider the following problem:

$$V' = C \left( \left( K \nu t_*^{\frac{1-\theta}{\theta}} \right)^{\frac{\lambda}{\lambda - \nu}} V_\rho + \epsilon_\nu^\nu (\rho - \rho_0)^{\nu/(1-\nu)} \right), \quad \rho \in (\rho_0, R),$$

$$V(\rho_0) = 0, \quad V(R) = L$$

with the constant $L$ from condition (5.2). It is easy to check that this problem admits a solution of the form

$$V(\rho) = A(\rho - \rho_0)^{\nu/(1-\nu)}, \quad \text{with} \quad A = L(R - \rho_0)^{-1/(1-\nu)},$$

provided that the parameters $t_*$, $\epsilon_*$ satisfy the relation

$$A' = C \left( \left( K \nu t_*^{\frac{1-\theta}{\theta}} \right)^{\frac{\lambda}{\lambda - \nu}} \frac{A}{1 - \nu} + \epsilon_\nu^\nu \right). \quad (5.19)$$

For every $A$ there always exist $t_* > 0$ and $\epsilon_* > 0$ such that (5.19) is fulfilled.

It is proved in [2, Ch.1, Lemma 2.4] that $V(\rho)$ majorates $E(\rho, t_\ast)$ on the interval $[\rho_0, R]$. Since $E(\rho, t)$ is monotone increasing in $\rho$ and $t$, it is necessary that $E(\rho, t) = 0$ for $\rho \in (0, \rho_0]$ and $t \in (0, t_\ast]$. It follows that $u = \text{const}$ in $Q_{\rho_0, T}$, whence $u = 0$ therein because $u_0 = 0$ in $B_{\rho_0}$ by assumption. \square

**Remark 5.1.** Condition (5.19) connects the three characteristic parameters of the problem: the total energy $L$, the waiting time $t_\ast$ and the threshold value of the source intensity $\epsilon_*$. For this reason, given an arbitrary intensity $0 < \epsilon_* < \infty$, the effect of waiting time of the solution can be provided by an appropriate choice of $t_\ast$ and $L$. 


6. Explicit solutions and failure of the maximum principle

6.1. Failure of the maximum and comparison principles in the case $p = 2$

The following simple example shows that in general the solutions of the linear equation (1.1) do not obey the maximum and comparison principles. Let us consider the problem

$$\begin{align*}
    u_t - \Delta u &= \int_0^t g(t-s)\Delta u \, ds \quad \text{in } Q = \Omega \times (0, T), \\
    u &= 0 \quad \text{on } \partial \Omega, \quad u(x, 0) = c_0 \psi(x) \quad \text{in } \Omega,
\end{align*}$$

(6.1)

where $\psi(x)$ is a nonnegative and bounded eigenfunction of the Dirichlet problem for the Laplace operator:

$$-\Delta \psi = \lambda \psi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial \Omega, \quad \lambda > 0.$$

The unique weak solution of problem (6.1) has the form $u = c(t)\psi(x)$ with the coefficient $c(t)$ defined from the conditions

$$c'(t) + \lambda c(t) = -\lambda \int_0^t g(t-s)c(s) \, ds, \quad c(0) = c_0. \quad (6.2)$$

We assume that the memory kernel has the form $g(s) = \mu e^{-\alpha s}$ with $\mu = \text{const}$ and $\alpha = \text{const} > 0$, so that $g'(t-s) = -\alpha g(t-s)$. Let us show now that for different choices of the parameters $\mu$ and $\alpha$ problem (6.1) with the same nonnegative and bounded initial function $u_0(x) = c_0 \psi(x)$, $c_0 = \text{const} > 0$, admits unbounded solutions, solutions that change the sign or oscillate around zero as $t \to \infty$. Differentiating the integro-differential equation (6.1) with respect to $t$ and plugging (6.2) we arrive at the 2nd-order ODE for the coefficient $c(t)$

$$c''(t) + (\lambda + \alpha)c'(t) + \lambda(\mu + \alpha)c(t) = 0, \quad c(0) = c_0, \quad c'(0) = -\lambda c_0. \quad (6.3)$$

The unique solution of this equation is presented explicitly.

(a) $(\lambda + \alpha)^2 > 4\lambda(\mu + \alpha)$. The characteristic equation has two different real roots

$$s_{1,2} = -\frac{\lambda + \alpha}{2} \pm \frac{1}{2} \sqrt{(\lambda + \alpha)^2 - 4\lambda(\mu + \alpha)},$$

and the solution is given by the formula

$$c(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t}, \quad a_i = \text{const}.$$  

There are three possibilities:

(1) if $\mu + \alpha < 0$, then $c(t)$ is unbounded as $t \to \infty$;

(2) if $\mu + \alpha = 0$, then $c(t) = a_1 + a_2 e^{-(\lambda+\alpha)t} \to a_1$ as $t \to \infty$;

(3) if $\mu + \alpha > 0$, then $c(t)$ decays exponentially as $t \to \infty$.

(b) $(\lambda + \alpha)^2 = 4\lambda(\mu + \alpha)$. In this case the characteristic equation has only one real root $s = -\frac{\lambda + \alpha}{2}$ and

$$c(t) = c_0 e^{-\frac{\lambda + \alpha}{2} t} - \frac{c_0}{2}(\lambda - \alpha) t e^{-\frac{\lambda + \alpha}{2} t} = c_0 \left(1 - (\lambda - \alpha) \frac{t}{2}\right) e^{-\frac{\lambda + \alpha}{2} t}.$$  

If $\alpha < \lambda$ this solution is positive for $t < 2/((\lambda - \alpha)$ and negative if $t > 2/((\lambda - \alpha))$. 

(c) \((\lambda + \alpha)^2 < 4\lambda(\mu + \alpha)\). The roots of the characteristic equation are complex numbers
\[
s_{1,2} = -\frac{\lambda + \alpha}{2} \pm iR, \quad R = \frac{1}{2}\sqrt{4\lambda(\mu + \alpha) - (\lambda + \alpha)^2},
\]
and the solution of problem (6.3) is represented in the form
\[
c(t) = c_0 e^{-\frac{\lambda + \alpha}{2} t} \cos(Rt) - \frac{c_0}{2R}(\lambda - \alpha)e^{-\frac{\lambda + \alpha}{2} t} \sin(Rt).
\]
This solution decays exponentially but does not preserve the sign if, say, \(\alpha = \lambda\) and \(\mu\) satisfies the inequality \(\mu > 0\).

6.2. Explicit solutions with exponential decay. The case \(p \in (1,2)\)

As was noticed in the introduction, the solutions of Eq. (1.1) with nonlocal memory do not inherit the property of time localization typical for the solutions of Eq. (1.4) with \(p \in (1,2)\). Let us consider the problem with one independent space variable
\[
\begin{aligned}
& u_t - (|u_x|^p - 2u_x)_x = \int_0^t g(t-s)(|u_x(x,s)|^p - 2u_x(x,s))_x \, ds + Lu \quad \text{in } Q = (-1,1) \times (0,T), \\
& u(\pm 1, t) = 0, \quad u(x,0) = u_0 \quad \text{in } (-1,1), \quad p \in (1,2), \quad L = \text{const}.
\end{aligned}
\]
The memory kernel has the form \(g(s) = -M e^{-\alpha s}\) with given positive constants \(M\) and \(\alpha\). Let us construct a solution with the method of separation of variables. Set \(u(x,t) = \theta(t)X(x)\) with the new unknowns \(\theta(t)\) and \(X(x)\). Let us take for \(X\) a solution of the problem
\[
\begin{aligned}
& (|X'|^{p-2}X')' = -\lambda X \quad \text{in } (-1,1), \\
& X(\pm 1) = 0
\end{aligned}
\]
with a positive constant \(\lambda > 0\), which is still arbitrary. It is straightforward to check that the integral
\[
\int_{Y(x)} \frac{dz}{(D^2 - z^2)^{\frac{p}{2}}} = \left(\frac{\lambda p}{2(p-1)}\right)^{\frac{1}{p}} x, \quad x \in [0,1], \quad D = \text{const} > 0
\]
defines a function \(Y(x)\) which solves Eq. (6.5) in \((0,1)\), is strictly monotone decreasing and satisfies the boundary conditions \(Y(0) = D, Y'(0) = 0\). The constant \(D\) can be chosen so that \(Y(1) = 0\):
\[
D^\frac{p}{2} \int_0^1 \frac{dy}{(1-y^2)^{\frac{p}{2}}} = \int_0^D \frac{dz}{(D^2 - z^2)^{\frac{p}{2}}} = \left(\frac{\lambda p}{2(p-1)}\right)^{\frac{1}{p}}.
\]
A solution of problem (6.5) is defined by the even continuation of \(Y(x)\) to the interval \((-1,1)\). Substituting \(\theta X\) into (6.4), using (6.5) and simplifying, for \(\theta(t)\) we obtain the ordinary integro-differential equation
\[
\mathcal{L}\theta(t) \equiv \theta'(t) + \lambda|\theta(t)|^{p-2}\theta(t) - \lambda M \int_0^t e^{-\alpha(t-s)}|\theta(s)|^{p-2}\theta(s) \, ds - L\theta(t) = 0, \quad \theta(0) = \theta_0 > 0.
\]
The function \(\theta(t)\) is sought in the form \(\theta(t) = \theta_0 e^{-\gamma t}\) with \(\gamma > 0\) to be defined. A direct computation leads to the equality
\[
\mathcal{L}\theta(t) = -\theta_0 (\gamma + L) e^{-\gamma t} + \frac{\lambda M \theta_0^{p-1}}{\alpha - \gamma(p-1)} e^{-\alpha t} + \lambda \theta_0^{p-1} \left(1 - \frac{M}{\alpha - \gamma(p-1)}\right) e^{-\gamma(p-1)t}.
\]
If we claim \(\gamma = \alpha\), this equality transforms into
\[
\mathcal{L}\theta(t) = \left(-\theta_0 (\alpha + L) + \frac{\lambda M \theta_0^{p-1}}{\alpha(2-p)}\right) e^{-\alpha t} + \lambda \theta_0^{p-1} \left(1 - \frac{M}{\alpha(2-p)}\right) e^{-\alpha(p-1)t},
\]
so \( L \theta(t) = 0 \) if \( L > -\alpha \) and

\[
\frac{M}{\alpha(2 - p)} = 1, \quad \lambda = \theta_0(\alpha + L)\theta_0^{1-p}.
\]

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