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On a pseudoparabolic problem with constraint

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Abstract
This work deals with the study to a nonlinear degenerated pseudoparabolic
problem. Arising from the modelling of sedimentary basins formation, the
equation degenerates in order to take implicitly into account a constraint
on the time-derivative of the unknown. An existence result of a solution
with an adapted compactness result and qualitative properties of the so-
lutions are proposed (localization, finite speed of propagation, etc.).
Keywords: Pseudoparabolic, compactness, finite speed propagation.

1 Introduction
The original motivation of this study is the mathematical analysis of general
models arising from geological basin formation. The initial model has been de-
veloped by the Institut Français du Pétrole and it takes into account sedimenta-
tion, transport and accumulation, erosion phenomena at large scales in time and
space. The main feature of these models is characterized by a constraint on the
time-derivative of the solution. This constraint leads us to consider an original
class of conservation laws, a priori parabolic or pseudoparabolic, revealing some
hyperbolic comportment because of a diffusive coefficient that depends on the
time-derivative of the solution. A more precise description of these models have
been exposed, on the one hand, for the multilithological case, by R. Eymard et
al. [?] and V. Gervais et al. [?]; on the second hand, by S. N. Antontsev et al.
[?, ?], G. Gagneux et al. [?, ?] and G. Vallet [?] for the mathematical aspect

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of the monolithological case and R. Eymard and al. [?] for a theoretical and numerical approach of an inverse problem.

For a brief presentation of the model, let us consider a sedimentary basin whose base, denoted by $\Omega$, is a fixed domain of $\mathbb{R}^d$ ($d = 1, 2$) with a Lipschitzian boundary $\Gamma$. The sediment height, denoted by $u$, naturally satisfies the mass balance equation $\partial_t u + \text{div} \{ \vec{q} \} = 0$ in $[0,T] \times \Omega$, where $\vec{q}$ is proportional to $\nabla [u + \tau \partial_t u]$ according to a Darcy-Barenblatt’s law (see C. Cuesta et al. [?]).

Following D. Granjeon [?], in the framework of a perfect physical equilibrium, the flux is considered to be proportional to the slope by writing that $\vec{q} = \vec{S} \propto -\nabla u$, i.e. $\tau = 0$. Taking into account a balancing of the slope leads us to consider $\vec{q} = \vec{S}_e$ with the first order kinetic equation $\vec{S}_e = \vec{S} + \tau \partial_t \vec{S}$, where $\tau > 0$ is time-scaled.

Moreover, in a sedimentary basin formation process, sediments must first be produced in situ by weathering effects prior to be transported by surfacing erosion. Thus, R. Eymard et al. [?] introduce a maximum erosion rate $E$ such that $-\partial_t u \leq E$ in $[0,T] \times \Omega$, where $E$ takes into account the composition, the structure and the age of the sediments. Then, the authors propose a sediment flux given by $-\vec{q} = -\lambda \nabla [u + \tau \partial_t u]$ where $\lambda$, $0 \leq \lambda \leq 1$, is playing the role of a flux limiter, with the relevant law of state in the form:

$$1 - \lambda \geq 0, \quad \partial_t u + E \geq 0, \quad (1 - \lambda) (\partial_t u + E) = 0 \text{ a.e. in } [0,T] \times \Omega.$$ 

Therefore, S. N. Antontsev et al. [?], G. Gagneux et al. [?] and G. Vallet [?] propose a conservative formulation that contains implicitly (see Remark ??) the above mathematical modelling of $\lambda$:

$$0 \in \partial_t u - \text{div} (H(\partial_t u + E) \nabla [u + \tau \partial_t u]) \text{ in } [0,T] \times \Omega,$$

i.e. $0 = \partial_t u - \text{div} (\lambda \nabla [u + \tau \partial_t u])$ where $\lambda \in H(\partial_t u + E)$ in $[0,T] \times \Omega$,

where $H$ denotes the maximal monotone graph of the function of Heaviside.

Existence and uniqueness results for a solution to the above differential inclusion are still open problems. Our purpose is to analyse a modified model where $H$ is replaced by a continuous function $a$, an approximation Yosida of $H$ for example.

An existence result of a solution to a nonlinear degenerate pseudoparabolic equation by the way of a compactness argument is presented, generalizing the first result of G. Gagneux and G. Vallet in [?]. Then, a qualitative study of the local hyperbolic behaviour, via finite speed propagation properties, of the problem is presented. This study points out the strong influence of the regularity of the function $a$, in the neighbourhood of $0^+$, on these descriptive properties.

### 1.1 Presentation of the model

Let us consider a Lipschitzian bounded domain $\Omega$, with boundary $\Gamma$ separated in two nontrivial distinct parts $\Gamma_s$ and $\Gamma_e$. For any positive $T$, let us denote $Q := [0,T] \times \Omega$. 

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Therefore, we are interested in the following pseudoparabolic problem:

\[ \partial_t u - \text{div}(a(\partial_t u + E)\nabla [u + \tau \partial_t u]) = 0 \quad \text{in} \quad Q, \tag{1} \]

with the initial height given by: \( u(0, x) = u_0(x), \ x \in \Omega, \)
and the following boundary conditions:

\[ -a(\partial_t u + E)\nabla [u + \tau \partial_t u] \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_\varepsilon, \]
\[ -a(\partial_t u + E)\nabla [u + \tau \partial_t u] \cdot \mathbf{n} \in \beta(\partial_t u) \quad \text{on} \quad \Gamma_s. \]

\( \beta \) is assumed to be the real graph of a maximal monotone operator (cf. J.-L. Lions [?] p.422 and references therein) such that:

i) \( 0 \in \beta(0), \ \forall x, y \in \mathbb{R}, \ \forall (\xi, \theta) \in \beta(x) \times \beta(y), \ (\xi - \theta)(x - y) \geq 0 \) (2)

(ii) There exists \( C_1, C_2 \) in \( \mathbb{R}^{+} \) satisfying the growth control condition:

\[ \forall x \in \mathbb{R}, \ \forall \theta \in \beta(x), \ C_1 \theta^2 \leq x \theta + C_2. \tag{3} \]

(iii) There exists a sequence \( (f_k)_{k \in \mathbb{N}} \subset C^1(\mathbb{R}) \), compatible with the growth control, such that: \( f_k(0) = 0, \ f'_k > 0 \) and

\[ f_k(\pm \infty) = \pm \infty \quad \text{and} \quad f_k^{-1} \rightarrow \beta^{-1} \quad \text{uniformly on any compact set.} \tag{4} \]

In the sequel \( \tau > 0, E : [0, T] \rightarrow \mathbb{R} \) is a non negative continuous function and \( a \) is a continuous function defined on \( \mathbb{R} \) that satisfies

\[ 0 \leq a \leq M, \quad a(0) = 0, \quad \forall x > 0, \ a(x) > 0 \quad \text{and} \quad a(-x) = 0 \quad \text{by extention} \]

in order to have implicitly \( \partial_t u + E \geq 0 \) a.e. in \( Q \) (Cf. Rmq ??-ii).

For any real \( x \), denote \( A(x) = \int_0^x a(s) \ ds \), and for technical reasons (the motivation is to derive various ways of approximating the Heaviside graph by Yosida regularized functions), one assumes that \( a \circ A^{-1} \) has got on \( \mathbb{R}^{+} \) a continuity modulus which square is an Osgood function; for simplicity, we consider the case where \( a \circ A^{-1} \) is Hölder continuous with exponent \( \frac{1}{2} \). Note that if \( a(x) \sim cx^\alpha \) then the hypothesis on \( a \circ A^{-1} \) imply that \( \alpha \geq 1 \).

1.2 Definition of a solution

By considering weak solutions, one is looking for \( u \) a priori in \( H^1(Q) \), with \( \partial_t u \) in \( L^2(0, T; H^1(\Omega)) \), such that for any \( v \) in \( H^1(\Omega) \) and for a.e. \( t \) in \( [0, T] \),

\[ 0 \in \int_{\Omega} \{ \partial_t uv + a(\partial_t u + E)\nabla [u + \tau \partial_t u] \nabla v \} \ dx + \int_{\Gamma_s} \beta(\partial_t u) v \ d\sigma. \tag{5} \]

i.e. there exists a measurable function \( \theta \) in \( \beta(\partial_t u) \) such that

\[ \int_{\Omega} \{ \partial_t uv + a(\partial_t u + E)\nabla [u + \tau \partial_t u] \nabla v \} \ dx + \int_{\Gamma_s} \theta v \ d\sigma = 0 \]

together with the initial condition \( u_{|t=0} = u_0 \) a.e. in \( \Omega \).
Remark 1  1. The problem can be written in the non-classical form:

\[
\begin{aligned}
\partial_t u - \tau \Delta A(\partial_t u + E) - \text{div}(a(\partial_t u + E) \nabla u) &= 0 \quad \text{in } Q, \\
\partial_t u + E &\geq 0 \quad \text{in } Q \text{ and } ]0,T[\times\Gamma, \\
-\tau \partial_n A(\partial_t u + E) - a(\partial_t u + E) \partial_n u &= 0 \quad \text{on } ]0,T[\times\Gamma, \\
-\tau \partial_n A(\partial_t u + E) - a(\partial_t u + E) \partial_n u &\in \beta(\partial_t u) \quad \text{on } ]0,T[\times\Gamma, \\
\varphi|_{t=0} &= u_0 \quad \text{a.e. in } \Omega.
\end{aligned}
\]

2. Given by extension that \(a(x) = 0\) for any negative \(x\), the suitable test function \(v = -(\partial_t u + E)^-\) leads, for a.e. \(t\) in \([0,T]\), to:

\[
\int_\Omega [(\partial_t u + E)^-]^2 dx - \int_{\Gamma_\tau} \theta (\partial_t u + E)^- d\sigma = -\int_\Omega (\partial_t u + E)^- E dx \leq 0
\]

where \(\theta \in \beta(\partial_t u)\). Since \(-\theta(\partial_t u + E) \geq 0\), one gets \(\partial_t u + E \geq 0\) a.e. in \(Q\).

3. If \(u_0 \in H^1(\Omega)\), then \(u\) belongs to \(L^\infty(0,T;H^1(\Omega))\). Let us consider \(\varepsilon > 0\) and \(v_\varepsilon = \int_0^{\varepsilon} \frac{ds}{a(s + E) + \varepsilon}\).

Then, the variational formulation gives

\[
0 = \int_\Omega \left( \frac{\partial_t u}{a(s + E) + \varepsilon} \right) dx - \int_{\Gamma_\tau} \theta \frac{\partial_t u + E}{a(\partial_t u + E) + \varepsilon} \nabla[u + \tau \partial_t u] \nabla \partial_t u \right) dx \\
+ \int_{\Gamma_\tau} \theta \frac{\partial_t u}{a(s + E) + \varepsilon} ds
\]

where \(\theta \in \beta(\partial_t u)\).

Since \(\partial_t u \int_0^{\varepsilon} \frac{ds}{a(s + E) + \varepsilon} \geq \frac{1}{M + \varepsilon} |\partial_t u|^2\) and \(\theta \int_0^{\varepsilon} \frac{ds}{a(s + E) + \varepsilon} \geq 0\), one gets that

\[
0 \geq \int_\Omega \left( \frac{|\partial_t u|^2}{M + \varepsilon} + \frac{a(\partial_t u + E)}{a(\partial_t u + E) + \varepsilon} \nabla[u + \tau \partial_t u] \nabla \partial_t u \right) dx
\]

Note that, when \(\varepsilon \to 0^+\),

\[
\frac{a(\partial_t u + E)}{a(\partial_t u + E) + \varepsilon} \nabla[u + \sigma \partial_t u] \nabla \partial_t u \to \nabla[u + \tau \partial_t u] \nabla \partial_t u \quad \text{a.e. in } \Omega,
\]

since \(\nabla[\partial_t u + E] = \nabla \partial_t u = 0\) a.e. in the set \(\{\partial_t u + E\} = 0\) and since \(\partial_t u + E \geq 0\).

Moreover, given that

\[
\left| \frac{a(\partial_t u + E)}{a(\partial_t u + E) + \varepsilon} \nabla[u + \sigma \partial_t u] \nabla \partial_t u \right| \leq |\nabla[u + \tau \partial_t u] \nabla \partial_t u| \quad \text{a.e. in } \Omega,
\]

the theorem of dominated convergence leads to:

\[
\frac{1}{M} \|\partial_t u\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla[u + \tau \partial_t u] \nabla \partial_t u| dx \leq 0.
\]
Thus,
\[ \frac{1}{M} \| \partial_t u \|^2_{L^2(\Omega)} + \frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(\Omega)} + \tau \| \nabla \partial_t u \|^2_{L^2(\Omega)} \leq 0 \]
and for a.e. \( t \) in \( ]0, T[ \),
\[ \frac{1}{M} \| \partial_t u \|^2_{L^2(0,t, L^2(\Omega))} + \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\Omega)} + \tau \| \nabla \partial_t u \|^2_{L^2(0,t, L^2(\Omega))} \]
\[ \leq \frac{1}{2} \| \nabla u_0 \|^2_{L^2(\Omega)}. \]
Since for any \( t \),
\[ \| u(t) \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} + \sqrt{t} \| \partial_t u \|_{L^2(0,t, L^2(\Omega))} \]
the result holds.

Note that this is also true for any \( t \) since \( u \in C_a([0, T], H^1(\Omega)) \), i.e. \( u(t) \) exists in \( H^1(\Omega) \) for any \( t \).
Moreover, one may remark that \( t \mapsto \| \nabla u(t) \|_{L^2(\Omega)} \) is a non increasing function.

4. If \( u_0 \in H^1(\Omega) \), then \( \partial_t u \) belongs to \( L^\infty(0, T; H^1(\Omega)) \).

Let us consider again inequality (??) to obtain with the above estimations
\[ \frac{1}{M} \| \partial_t u \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \partial_t u \|^2_{L^2(\Omega)} \leq \frac{1}{2\tau} \| \nabla u_0 \|^2_{L^2(\Omega)} \]
and conclude.

1.3 Existence of a strong solution

1.3.1 A uniqueness lemma

The key to understanding compactness properties in the regularizing procedure is the following assertion:

**Lemma 1** Let us consider \( \kappa \) in \( H^1(\Omega) \), \( E \) a real number, \( \beta \) a monotone graph (not necessary maximal) and \( b \) an essentially bounded nonnegative continuous function such that
\[ \forall x, y \in \mathbb{R}, \ |b(x) - b(y)| \leq c |B(x) - B(y)|^{\frac{1}{2}} \] where \( B(x) = \int_0^x b(s) \, ds. \)

Then, there exists at most one solution \( u \) in \( H^1(\Omega) \) such that for any \( v \) in \( H^1(\Omega) \),
\[ 0 \in \int_\Omega \{ uv + b(w + E)\nabla[\kappa + w] \nabla v \} \, dx + \int_{\Gamma_s} \beta(w) v \, ds. \quad (7) \]
Proof. If one denotes by \( w_1 \) and \( w_2 \) two admissible solutions. Let us set \( \xi = B(w_1 + E) - B(w_2 + E) \) and, for a given \( \mu > 0 \), \( p_\mu(t) = \begin{cases} 1 & \text{if } t \geq \mu, \\ \ln \left( \frac{t}{\mu} \right) & \text{if } t \in \left[ \frac{\mu}{e}, \mu \right], \\ 0 & \text{if } t \leq \frac{\mu}{e}. \end{cases} \)

\( p_\mu \) is a non-increasing lipschitzian function with a lipschitz-constant equal to \( \frac{\mu}{e} \).

Therefore, \( v = p_\mu[\xi] \) is a suitable test function and it comes

\[
0 = \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} \frac{1}{\xi} |\nabla \xi|^2 dx + \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} \frac{b(w_1 + E) - b(w_2 + E)}{\xi} \nabla \kappa \nabla \xi \, dx \\
+ \int_{\Omega} (w_1 - w_2) p_\mu(\xi) \, dx + \int_{\Gamma} [\theta_1 - \theta_2] p_\mu(\xi) \, d\sigma,
\]

where \( \theta_i \in \beta(w_i) \).

Since

\[
(\theta_1 - \theta_2) p_\mu(\xi) = (\theta_1 - \theta_2)(w_1 - w_2) \frac{p_\mu(\xi)}{(w_1 + E) - (w_2 + E)} \geq 0,
\]

so,

\[
\int_{\Omega} (w_1 - w_2) p_\mu(\xi) \, dx + \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} \frac{1}{\xi} |\nabla \xi|^2 dx \\
\leq \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} \frac{c \sqrt{\xi}}{\xi} |\nabla \kappa \nabla \xi| \, dx \\
\leq C \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} |\nabla \kappa|^2 dx + \int_{\Omega \cap \{ \frac{\mu}{e} \leq \xi \leq \mu \}} \frac{1}{2\xi} |\nabla \xi|^2 dx
\]

and the solution is unique by considering the limits when \( \mu \) goes to \( 0^+ \). \( \square \)

1.4 The univoque case

Let us assume in this section that \( \beta \in C^1(\mathbb{R}) \) with \( \beta(0) = 0 \) and \( \beta' > 0 \).

1.4.1 Semi-discretized non degenerated processes

Let us consider \( h > 0, u_0 \in H^1(\Omega), E \geq 0 \) (let us assume \( E \) constant to simplify slightly the notations, \( E = 5m/Myr \) for example in R. Eymard et al. [?]) and for a given positive \( \alpha \), \( a_0 = \max(\alpha, a) \).

Proposition 1 There exists a unique \( u_\alpha \) in \( H^1(\Omega) \) such that for any \( v \) in \( H^1(\Omega) \),

\[
0 = \int_{\Gamma} \beta(\frac{u_\alpha - u_0}{h}) v \, d\sigma \\
+ \int_{\Omega} \left\{ \frac{u_\alpha - u_0}{h} v + a_0 \left( \frac{u_\alpha - u_0}{h} + E \right) \nabla u_\alpha + \frac{u_\alpha - u_0}{h} \nabla v \right\} \, dx.
\]
Proof. The demonstration is classical since if one denotes by \( w = u - u_0 \), the problem is lead to: find \( w \) in \( H^1(\Omega) \) such that, for any \( v \) in \( H^1(\Omega) \),

\[
\int_{\Omega} \{wv + (h + \tau)\nabla A_\alpha(w + E)\nabla v + a_\alpha(w + E)\nabla u_0 \nabla v\} \, dx + \int_{\Gamma_s} \beta(w)v \, d\sigma = 0
\]

where \( A_\alpha \) a bi-lipschitzian continuous function.

On the one hand, Schauder-Tikhonov fixed point theorem and the compactness\(^1\) of the trace operator from \( H^1(\Omega) \) into \( L^2(\Gamma) \) are used for the existence of a solution in the hilbertian separable framework. Firstly, the assertion is proved with \( \beta_n = \max(-n, \min(\beta, n)) \) where \( n \in \mathbb{N} \).

Then, thanks to \textit{a priori} estimates, one is able to pass to the limits with \( n \) towards \(+\infty\) since

\[
\int_{\Gamma_s} \beta_n(w) \int_0^w \frac{ds}{a_\alpha(s + E)} \, d\sigma \geq \int_{\Gamma_s} \beta_n(w) \frac{w}{M} \, d\sigma \geq C_1 \int_{\Gamma_s} \beta_n^2(w) \, d\sigma - C_2 \frac{M}{M} \text{mes}(\Gamma_s).
\]

On the other hand, the uniqueness of the solution comes from Lemma ?? with \( b = (h + \tau) a_\alpha, \kappa = \frac{u_0}{h + \tau} \) and the graph \( x \mapsto \{\beta(x)\} \).

1.4.2 Semi-discretized degenerated problem

Let us consider \( h > 0, u_0 \in H^1(\Omega), E \geq 0 \) and in the sequel, \( a(x) = 0 \) for any nonnegative real \( x \).

Proposition 2 There exists a unique \( u \) in \( H^1(\Omega) \) such that,

\[
\forall v \in H^1(\Omega), \quad 0 = \int_{\Omega} \{\frac{u - u_0}{h} v + a\frac{u - u_0}{h} + E\nabla|u + \tau \frac{u - u_0}{h} \nabla v|\} \, dx + \int_{\Gamma_s} \beta\left(\frac{u - u_0}{h}\right)v \, d\sigma. \tag{8}
\]

Moreover, \( \frac{u - u_0}{h} + E \geq 0 \) a.e. in \( \Omega \).

Proof. Note that the last inequality is obvious by using \( v = -(\frac{u - u_0}{h} + E)^- \).

Let us note \( w_\alpha = \frac{u - u_0}{h} \) and consider \( v = \int_0^{\frac{u - u_0}{h}} \frac{ds}{a_\alpha(s + E)} \) in the equation of proposition (??) to obtain

\[
0 \geq \int_{\Omega} \left\{\frac{u_\alpha - u_0}{h} \int_0^{\frac{u_\alpha - u_0}{h}} \frac{ds}{a_\alpha(s + E)} + \nabla[u_\alpha + \tau \frac{u_\alpha - u_0}{h} \frac{u_\alpha - u_0}{h} \nabla v]\right\} \, dx + \int_{\Gamma_s} \beta\left(\frac{u_\alpha - u_0}{h}\right) \int_0^{\frac{u_\alpha - u_0}{h}} \frac{ds}{a_\alpha(s + E)} \, d\sigma.
\]

\(^1\)Remind that the embedding of \( H^1(\Omega) \) into \( H^s(\Omega) \) is compact for any \( \frac{1}{2} < s < 1 \) and the trace operator of \( H^s(\Omega) \) into \( L^2(\Gamma) \) is continuous.
Since \( u_\alpha - u_0 \rightarrow_\alpha \int_0^\infty \frac{ds}{a_\alpha(s+E)} \geq \frac{1}{M} \left\| u_\alpha - u_0 \right\|^2 \) and

\[
\int_{\Gamma_s} \beta\left( \frac{u_\alpha - u_0}{h} \right) \int_0^\infty \frac{ds}{a_\alpha(s+E)} \, d\sigma \geq \int_{\Gamma_s} \beta\left( \frac{u_\alpha - u_0}{h} \right) \frac{u_\alpha - u_0}{hM} \, d\sigma
\]

\[
\geq C_1 \int_{\Gamma_s} \beta^2\left( \frac{u_\alpha - u_0}{h} \right) \, d\sigma - \frac{C_2}{M} \text{mes}(\Gamma_s),
\]

one gets

\[
\frac{C_2}{M} \text{mes}(\Gamma_s) \geq \frac{1}{2h} \left( \| u_\alpha \|_{H^1(\Omega)}^2 + \| u_\alpha - u_0 \|_{H^1(\Omega)}^2 - \| u_0 \|_{H^1(\Omega)}^2 \right) + \frac{1}{M} \| u_\alpha - u_0 \|_{L^2(\Omega)}^2 + \tau \| u_\alpha - u_0 \|_{H^1(\Omega)}^2 + C_1 \| \beta^2(u_\alpha - u_0) \|_{L^2(\Gamma_s)}.
\]

Therefore, \( \left( \frac{u_\alpha - u_0}{h} \right)_\alpha \) and \( (u_\alpha)_\alpha \) are bounded sequences in \( H^1(\Omega) \) and passing to limits is possible since up to a sub-sequence \( \frac{u_\alpha - u_0}{h} \) converges a.e. in \( \Omega \) and \( \Gamma \).

At last, the uniqueness can be proved by using Lemma ?? with \( b = (h + \tau) a \), \( \kappa = \frac{u_0}{h + \tau} \) and the graph \( x \mapsto \{ \beta(x) \} \).

\[ \square \]

### 1.4.3 Semi-discretized degenerated differential inclusion

**Proposition 3** There exists a pair \((u, \lambda)\) in \( H^1(\Omega) \times L^\infty(\Omega) \) such that \( \lambda \in H(\frac{u - u_0}{h} + E) \) where \( H \) is the maximal monotone graph of the function of Heaviside, satisfying for any \( v \) in \( H^1(\Omega) \),

\[
\int_{\Omega} \left\{ \frac{u - u_0}{h} v + \lambda \nabla [u + \tau \frac{u - u_0}{h}] \frac{\nabla v}{\nabla} \right\} \, dx + \int_{\Gamma_s} \beta\left( \frac{u - u_0}{h} \right) v \, d\sigma = 0. \tag{10}
\]

**Proof.** Let us assume in this section that for any given positive \( \varepsilon \), \( a = a_\varepsilon = \max(0, \min(1, \frac{1}{2} I d)) \).

The corresponding solutions are denoted by \( u_\varepsilon \) and thanks to the above inequality, a sub-sequence still indexed by \( \varepsilon \) can be extracted, such that \( u_\varepsilon \) converges weakly in \( H^1(\Omega) \) towards \( u \), strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \) and \( \Gamma \). Moreover, \( \frac{u - u_0}{h} + E \geq 0 \) a.e. in \( \Omega \).

Furthermore, \( A_\varepsilon \) converges uniformly towards \((\cdot)^+ \) and then \( A_\varepsilon(\frac{u - u_0}{h} + E) \) converges weakly in \( H^1(\Omega) \) towards \( \frac{u - u_0}{h} + E \).

Up to a new sub-sequence, let us denote by \( \lambda \) the weak--* limit in \( L^\infty(\Omega) \) of \( a_\varepsilon(\frac{u - u_0}{h} + E) \) and remark that inevitably \( \lambda = 1 \) a.e. in \( \{ \frac{u - u_0}{h} + E > 0 \} \) i.e. \( \lambda \in H(\frac{u - u_0}{h} + E) \). Since for any \( v \) in \( H^1(\Omega) \),

\[
\int_{\Omega} \left( \frac{u_\varepsilon - u_0}{h} v + (h + \tau) \nabla A_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E) \nabla v + a_\varepsilon(\frac{u_\varepsilon - u_0}{h} + E) \nabla u_0 \nabla v \right) \, dx + \int_{\Gamma_s} \beta\left( \frac{u_\varepsilon - u_0}{h} \right) v \, d\sigma = 0,
\]

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passing to the limits leads to

\[ \forall v \in H^1(\Omega), \quad 0 = \int_\Omega \frac{u-u_0}{h} v \, dx + \int_\Omega (h+\tau)v \frac{u-u_0}{h} \nabla v + \lambda \nabla u_0 \nabla v \, dx \]

and to

\[ \forall v \in H^1(\Omega), \quad \int_\Omega \left\{ \frac{u-u_0}{h} - \frac{u-u_0}{h} \nabla v \right\} \, dx + \int_{\Gamma_s} \beta \left( \frac{u-u_0}{h} \right) v \, d\sigma = 0, \]

since \( \nabla \frac{u-u_0}{h} = 0 \) a.e. in \( \{ \lambda \neq 1 \} \).

\[ \square \]

1.4.4 Existence of a solution

Inductively, the following result can be proved:

Let us consider \( N \in \mathbb{N}^* \) with \( h = \frac{T}{N}, u_0 \in H^1(\Omega) \) and \( E_k \geq 0 \) for any integer \( k \).

**Proposition 4** For \( u^0 = u_0 \), there exists a unique sequence \( (u^k)_k \) in \( H^1(\Omega) \) such that for any \( v \) in \( H^1(\Omega) \),

\[ 0 = \int_\Omega \left\{ \frac{u^{k+1}-u^k}{h} v + a\left( \frac{u^{k+1}-u^k}{h} + E_k \right) \nabla \left[ u^{k+1} + \tau \frac{u^{k+1}-u^k}{h} \right] \nabla v \right\} \, dx \]

\[ + \int_{\Gamma_s} \beta \left( \frac{u^{k+1}-u^k}{h} \right) v \, d\sigma. \]

Moreover, \( \frac{u^{k+1}-u^k}{h} + E_k \geq 0 \) a.e. in \( \Omega \), and

\[
\frac{1}{M} \left| \frac{u^{k+1}-u^k}{h} \right|_{L^2(\Omega)}^2 + \frac{\tau}{h} \left| \frac{u^{k+1}-u^k}{h} \right|_{H^1(\Omega)}^2 + \frac{1}{2h} \left| u^{k+1} \right|_{H^1(\Omega)}^2 + \| u^{k+1} - u^k \|_{H^1(\Omega)}^2 - \| u^k \|_{H^1(\Omega)}^2 \leq 0. \quad (11)
\]

A priori estimations leading to the main result

In order to prove this result, for any sequence \( (v_k)_k \subset L^2(\Omega) \), let us note in the sequel

\[ v^h = \sum_{k=0}^{N-1} [v^k]_{kh,(k+1)h} \quad \text{and} \quad \tilde{v}^h = \sum_{k=0}^{N-1} \left[ \frac{v^{k+1}-v^k}{h} (t-kh) + v^k \right]_{kh,(k+1)h}. \]

**Lemma 2** \( (u^h)_h \) and \( (\tilde{u}^h)_h \) are bounded sequences in \( L^\infty(0,T,H^1(\Omega)) \) and \( \partial_t \tilde{u}^h \) is a bounded sequence in \( L^\infty(0,T,H^1(\Omega)) \) and then in \( L^\infty(0,T,L^2(\Gamma_s)) \).
Proof. This result comes from (??) for the first part and since
\[ 2\nabla (u^{k+1} + \tau \frac{u^{k+1} - u^k}{h}) \nabla \frac{u^{k+1} - u^k}{h} \geq \tau |\nabla \frac{u^{k+1} - u^k}{h}|^2 - \frac{1}{\tau} |\nabla u^{k+1}|^2 \]
implies that
\[ \frac{1}{M} \left| \frac{u^{k+1} - u^k}{h} \right|^2_{L^2(\Omega)} + \frac{\tau}{2} \left| \frac{u^{k+1} - u^k}{h} \right|^2_{H^1(\Omega)} \leq \frac{1}{2\tau} |u^{k+1}|^2_{H^1(\Omega)}. \]
\[ \Box \]

Existence of a solution
Our aim is to prove the following result, under the hypothesis of lemma ??:

**Proposition 5** There exists u in \( H^1(Q) \) with \( \partial_t u \) in \( L^2(0, T, H^1(\Omega)) \) such that for any \( v \) in \( H^1(\Omega) \) and for a.e. \( t \) in \( [0, T] \),
\[ \int \{ \partial_t uv + a(\partial_t u + E) \nabla [u + \tau \partial_t u \nabla v] \} \ dx + \int_{\Gamma} \beta(\partial_t u)v \ d\sigma = 0. \]
Moreover, \( u \) belongs to \( W^{1,\infty}(0, T, H^1(\Omega)) \).

Proof. Since \( (\tilde{u}^h)_h \) is bounded in \( H^1(0, T, H^1(\Omega)) \), there exists a sub-sequence, still indexed by \( h \) such that for any \( t \), \( \tilde{u}^h(t) \rightarrow u(t) \) in \( H^1(\Omega) \).
Moreover, if \( t \in [kh, (k+1)h] \), \( ||u^h(t) - \tilde{u}^h(t)||_{H^1(\Omega)} = ||\tilde{u}^h(kh) - \tilde{u}^h(t)||_{H^1(\Omega)} \leq \int_{kh}^{(k+1)h} ||\partial_t \tilde{u}^h(s)||_{H^1(\Omega)} ds \leq C h. \)
Then, for any \( t \), \( u^h(t) \rightarrow u(t) \) in \( H^1(\Omega) \).
Since \( \partial_t \tilde{u}^h \) is bounded in \( L^\infty(0, T, H^1(\Omega)) \), it follows that there exists a measurable set \( Z \subset [0, T] \) with \( \mathcal{L}(\{0, T\} \setminus Z) = 0 \), such that for any \( t \) in \( Z \), \( \partial_t \tilde{u}^h(t) \) is a bounded sequence in \( H^1(\Omega) \).
Therefore, up to a sub-sequence indexed by \( h_t \), \( \partial_t \tilde{u}^h \rightarrow \xi(t) \) in \( H^1(\Omega) \), strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \) with \( \xi(t) + E(t) \geq 0 \) a.e. in \( \Omega \), strongly in \( L^2(\Gamma) \) and a.e. in \( \Gamma \) by using the compactness of the trace operator from \( H^1(\Omega) \) into \( L^2(\Gamma) \).
Let us note that for any \( t \) in \( [kh, (k+1)h], \forall v \in H^1(\Omega), \)
\[ 0 = \int_{\Omega} \{ \partial_t \tilde{u}^h v + a(\partial_t \tilde{u}^h + E)v \nabla [u^h + \tau \partial_t \tilde{u}^h \nabla v] \} \ dx \quad (12) \]
\[ + \int_{\Gamma} \beta(\partial_t \tilde{u}^h)v \ d\sigma. \]
Given that, \( a(\partial_t \tilde{u}^h + E) \nabla v \) converges towards \( a(\xi(t) + E) \nabla v \) in \( L^2(\Omega) \) and that \( \nabla [u^h + \tau \partial_t \tilde{u}^h] \) converges weakly towards \( \nabla [u(t) + \tau \xi(t)] \) in \( L^2(\Omega) \), and thanks to the hypothesis on \( \beta, \xi(t) \) is a solution to the problem: at time \( t \), find \( w \) in \( H^1(\Omega) \) with \( w + E(t) \geq 0 \) a.e. in \( \Omega \), such that for any \( v \) in \( H^1(\Omega), \)
\[ \int_{\Omega} \{ \partial_t w + a(w + E(t)) \nabla [u(t) + \tau w] \nabla v \} \ dx + \int_{\Gamma} \beta(w)v \ d\sigma = 0. \quad (13) \]
The function \( \xi : [0, T] \rightarrow H^1(\Omega) \) is weakly measurable (indeed, for any \( f \) in \((H^1)'(\Omega), t \mapsto < f, \xi(t) > \) is the limit of the sequence of measurable functions \( t \mapsto < f, \partial_t \tilde{u}^b(t) > \)) and consequently \( \xi \) is a measurable function thanks to Pettis theorem (K. Yosida [?], p. 131), since \( H^1(\Omega) \) is a separable set.

For any \( v \) in \( L^2(0, T, H^1(\Omega)) \), \((\partial_t \tilde{u}^b(t), v(t))\) converges a.e. in \([0, T]\) towards \((\xi(t), v(t))\). Moreover, \(||(\partial_t \tilde{u}^b(t), v(t))|| \leq C||v(t)||_{H^1(\Omega)}\) a.e. since the sequence \((\partial_t \tilde{u}^b)_h\) is bounded in \( L^\infty(0, T, H^1(\Omega))\).

Thus, the weak convergence in \( L^2(0, T, H^1(\Omega)) \) of \( \partial_t \tilde{u}^b \) towards \( \xi \) can be proved. And one gets that \( \xi = \partial_t u \).

At last, for \( t \) a.e. in \([0, T]\), passing to limits in (??) leads to:

\[
\int_Q \{\partial_t uv + a(\partial_t u + E)\nabla [u + \varepsilon \partial_t u] \nabla v\} \, dxdt + \int_{[0, T] \times \Gamma_s} \beta(\partial_t u) v \, d\sigma dt = 0 \quad (14)
\]

and to the existence of a solution.

\[\square\]

### 1.5 The multivoque case

Let us prove in this section that if \( \beta \) is the maximal monotone graph presented in the introduction, then

**Proposition 6** There exists \( u \) in \( W^{1, \infty}(0, T; H^1(\Omega)) \) such that for any \( v \) in \( H^1(\Omega) \) and for a.e. \( t \) in \([0, T]\),

\[
0 \in \int_{\Omega} \{\partial_t uv + a(\partial_t u + E)\nabla [u + \tau \partial_t u] \nabla v\} \, dx + \int_{\Gamma_s} \beta(\partial_t u) v \, d\sigma. \quad (15)
\]

**Proof.** Following J.-L. Lions idea in [?], p.422 et passim, one denotes by \( \beta_n = \max(-n, \min(f_n, n)) \).

Associated with this sequence, one has a sequence \((u_n)\) of solutions to (??). Of course, this sequence is bounded in \( H^1(0, T; H^1(\Omega)) \) and the same kind of demonstration can be done in order to prove the existence of a solution to (??).

Let us considering the notations of the previous section. Then, for any \( t \) in \( Z \) and for the sub-sequence indexed by \( n_t \) such that \( \partial_t u_{n_t} \rightarrow \xi(t) \) in \( H^1(\Omega) \), strongly in \( L^2(\Gamma) \) and a.e. in \( \Gamma \), one has for any \( v \) in \( H^1(\Omega) \),

\[
\int_{\Omega} \{\partial_t u_{n_t} v + a(\partial_t u_{n_t} + E)\nabla [u_{n_t} + \tau \partial_t u_{n_t}] \nabla v\} \, dx + \int_{\Gamma_s} \beta_{n_t}(\partial_t u_{n_t}) v \, d\sigma = 0.
\]

Therefore, one gets that \( \beta_{n_t}(\partial_t u_{n_t}) \) converges weakly in \( L^2(\Gamma_s) \) towards an element denoted by \( \phi \). The same kind of demonstration than the one given p. 424 by J.-L. Lions [?] leads to \( \phi = \beta^{-1}(\xi(t)) \) and proves that \( \xi(t) \) is a
solution to the problem: at time $t$, find $w$ in $H^1(\Omega)$ with $w + E(t) \geq 0$ a.e. in $\Omega$, such that for any $v$ in $H^1(\Omega)$,
\[
\int_{\Omega} \{w v + a(w + E(t)) \nabla u(t) + \tau w \nabla v\} \, dx + \int_{\Gamma} \beta(w)v \, d\sigma \geq 0. \tag{16}
\]

Then, one gets a result of existence of a solution in the same way, according to the proposition ??.

Now, we examine more carefully the sets of points at which equation (??) degenerates, i.e. the $\mathcal{L}^d$-measurable sets $\mathcal{N}_s = \{x \in \Omega, \partial_t u(s) + E = 0\}, \ s \in [0,T]$, when the function $a$ is smooth enough in the neighbourhood of $0^+$. We will see that for a.e. $s$ in $[0,T]$, $\mathcal{N}_s$, the sets where the weathering rate $E$ is maximum, are $\mathcal{L}^d$-negligible if and only if $E > 0$. In this sense, the case $E = 0$ (sedimentation process) is singular.

**Proposition 7** Assume\(^2\) that $\lim_{x \to 0^+} \frac{a(x)}{x} < +\infty$ then:

if $E = 0$, $u = u_0$ is the unique solution, else, in the relevant case, $E > 0$, for $s$ a.e. in $[0,T]$, $\mathcal{L}^n \{\partial_t u(s) + E = 0\} = 0$, i.e. $\partial_t u + E > 0$ a.e. in $\Omega$.

Moreover, if $\beta(E) \ominus [\alpha, \beta 0], \mathcal{N}_s \neq \emptyset$, for $s$ a.e. in $[0,T]$, $\mathcal{H}^{n-1} \{\partial_t u(s) + E = 0\} = 0$, i.e. $\partial_t u + E > 0$ a.e. in $\Gamma$.

**Proof.** Let us consider the test function $v = p_s(\partial_t u + E)$ in equation (??) where for any $x$, $p_s(x) = \max(0, \min(\frac{x}{s}, 1))$. Since $\beta$ is monotone, for any $\theta \in \beta(-E)$, one gets, for a.e. $t$ in $[0,T]$,
\[
E \int_{\Omega} p_s(\partial_t u + E) \, dx - \int_{\Gamma} \theta p_s(\partial_t u + E) \, d\sigma \geq \int_{\Omega} \{\partial_t u(\partial_t u + E)p_s(\partial_t u + E) + a(\partial_t u + E)\nabla u + \tau \partial_t u + E)\nabla p_s(\partial_t u + E)\} \, dx.
\]

Thanks to the hypothesis on $a$ and the Saks lemma, it comes
\[
E \int_{\Omega} \text{sgn}_0^+ (\partial_t u + E) \, dx - \int_{\Gamma} \theta \text{sgn}_0^+ (\partial_t u + E) \, d\sigma \geq \int_{\Omega} |\partial_t u + E| \, dx.
\]

On the one hand, if $E = 0$, since $0 \notin \beta(0)$, one gets $|\partial_t u| = 0$ a.e. and the only solution is the trivial one: $u(t, x) = u_0(x)$.

On the other hand, if $E > 0$, test function $v = 1 - p_s(\partial_t u + E)$ in equation (??) leads to
\[
E \int_{\Omega} \{1 - \text{sgn}_0^+ (\partial_t u + E)\} \, dx + \int_{\Gamma} \theta (\text{sgn}_0^+ (\partial_t u + E) - 1) \, d\sigma \leq \int_{\Omega} \partial_t u + E - |\partial_t u + E| \, dx.
\]

\(^2\)Note that if $a(x) \sim cx^\alpha$ then the hypothesis on $a \circ A^{-1}$ imply that $\alpha \geq 1$. 

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Since \( \theta \) is nonpositive and \( \partial_t u + E \) nonnegative, it comes \( E, \mathcal{L}^n \{ \partial_t u + E = 0 \} \leq 0 \) and the result holds. Moreover, if \( \theta \) can be chosen negative, the last part of the property holds too. □

1.6 Localization properties of solutions. Finite speed of propagation

In this section we analyze localization properties of weak solutions of considered problem following to ideas of the book S. N. Antonev et al. [2]. We start consider local properties of solutions to generalized equation

\[
\partial_t u - \partial_x \left( a(\partial_t u, t) \right) (\partial_x u + \tau \partial_{xt} u) = 0 \quad \text{in} \quad Q = [0, T] \times \Omega, \quad \Omega = [0, 1].
\]

(17)

We assume that there exists a local weak solution of equation (17) satisfying to the boundary and initial conditions

\[
a(\partial_t u, t) (\partial_x u + \tau \partial_{xt} u) \big|_{x=0} = 0, \quad u(x, 0) = u_0(x), \quad x \in [0, 1],
\]

where

\[
u_0(x) = 0, \quad x \in [0, \rho_0], \quad \rho_0 < 1.
\]

(18)

Assume that the function \( a(r, t) \) satisfies to conditions: \( \forall (r, t) \in \mathbb{R} \times \mathbb{R}^+ \)

\[
0 \leq a, \quad \frac{1}{C_a} |r|^{2-\alpha} \leq r \int_0^r \frac{ds}{a(s, t)}, \quad a(r, t) \left| \int_0^r \frac{ds}{a(s, t)} \right| \leq C_a |r|,
\]

(20)

with some constants \( C_a \) and \( \alpha \in ]0, 1[. \)

**Remark 2** It is easy to verify that the function \( a(s) = |s|^\alpha, \quad 0 < \alpha < 1 \), satisfies (17).

1.6.1 Energy relations

Let us introduce the function

\[
\psi_k(\rho, x) = \begin{cases} 
1 & \text{for } x > \rho + 1/k, \\
k(x - \rho) & \text{for } x \in [\rho, \rho + 1/k], \\
0 & \text{for } x < \rho, \quad k = 1, 2, 
\end{cases}
\]

Substituting \( v = \psi_k \left( \int_0^{\partial_t u} \frac{ds}{a(s, t)} \right) \) into integral identity (Remark 2.3) as the test -function, we have

\[
\int_0^1 \left[ \partial_t u \left( \int_0^{\partial_t u} \frac{ds}{a(s, t)} \right) + (\partial_x u + \tau \partial_{xt} u) \partial_{xt} u \right] \psi_k dx + k \int_0^{\rho + 1/k} I(s, t) ds = 0,
\]

(21)
where
\[ I(x,t) = a(\partial_t u(x,t), \left( \int_0^{\partial_x u(x,t)} \frac{ds}{a(s,t)} \right) (\partial_x u(x,t) + \tau \partial_{xt} u(x,t)). \]

It follows from the Lebesgue theorem that for a.e. \( \rho \in (0, \rho_0) \)
\[
\lim_{k \to \infty} k \int_0^{\rho+1/k} I(s,t)ds = I(\rho, t).
\]

Passing into (??) to the limit on \( k \to +\infty \), we come to the energy relation
\[
\int_0^{\rho} \left[ \partial_t u \left( \int_0^{\partial_x u} \frac{ds}{a(s,t)} \right) + (\partial_x u + \tau \partial_{xt} u) \partial_{xt} u \right] dx = -I(\rho, t).
\]

Integrating last relation with respect to \( t \), and repeating of arguments of Remark ??, we obtain the inequalities
\[
\int_0^{\rho} \left[ \partial_t u \left( \int_0^{\partial_x u} \frac{ds}{a(s,t)} \right) + (\partial_x u + \tau \partial_{xt} u) \partial_{xt} u \right] dx + \frac{1}{2} \int_0^{\rho} (\partial_x u)^2 dx \leq \int_0^t |I(\rho, s)|ds.
\]

Multiplying (??) by \( \tau \frac{\tau}{2} \), adding to (??) and using (??) and (??), we come to the energy inequality
\[
\int_0^t \int_0^{\rho} \left( |\partial_t u|^{2-\alpha} + (\partial_{xt} u)^2 \right) dxdt + \int_0^{\rho} \left( |\partial_t u|^{2-\alpha} + |\partial_x u(x,t)\|^2 + (\partial_{xt} u)^2 \right) dx 
\leq C(\tau) \left( J(\rho, t) + \int_0^t J(\rho, s) ds \right),
\]
where
\[
|I(\rho, t)| \leq C J(\rho, t) = C|\partial_x u(\rho, t)| (|\partial_x u(\rho, t)| + |\partial_{xt} u(\rho, t)|) = C (J_1 + J_2).
\]

We introduce the energy functions
\[
E(\rho, t) \equiv \int_0^t \int_0^{\rho} \left( |\partial_t u|^{2-\alpha} + (\partial_{xt} u)^2 \right) dxdt
\]
\[
b(\rho, t) \equiv \int_0^{\rho} \left( |\partial_t u|^{2-\alpha} + |\partial_x u(x,t)|^2 + (\partial_{xt} u)^2 \right) dx
\]
satisfying

\[ E_\rho = \frac{\partial E(\rho, t)}{\partial \rho} = \int_0^t (|\partial_t u|^{2-\alpha} + (\partial_x u)^2) \, dt > 0, \]

\[ \sup_{0 \leq t \leq t_0} E_\rho = \int_0^{t_0} (|\partial_t u|^{2-\alpha} + (\partial_x u)^2) \, dt = \frac{\partial (\sup_{0 \leq t \leq t_0} E(\rho, t))}{\partial \rho}, \]

\[ b_\rho = \frac{\partial b(\rho, t)}{\partial \rho} = (|\partial_t u|^{2-\alpha} + |\partial_x u(x, t)|^2 + (\partial_x u)^2) > 0. \]

### 1.6.2 Global estimate for energy functions

For \( \rho = 1 \) the inequalities (??), (??) take the forms

\[ \int_0^1 \left[ \partial_t u \left( \int_0^{\partial_t u} \frac{ds}{a(s, t)} \right) + \frac{\tau}{2} |\partial_x u|^2 \right] \, dx \leq \frac{1}{2\tau} \int_0^1 |\partial_x u|^2 \, dx, \]

\[ \int_0^t \int_0^1 \left[ \partial_t u \left( \int_0^{\partial_t u} \frac{ds}{a(s, t)} \right) + \tau |\partial_x u|^2 \right] \, dx \, dt \]

\[ + \frac{1}{2} \int_0^1 (\partial_x u)^2 \, dx \leq \frac{1}{2} \int_0^1 (\partial_x u_0)^2 \, dx, \]

and give to us the estimate

\[ E(\rho, t) + b(\rho, t) \leq C(C_a, \tau) \int_0^1 (\partial_x u_0)^2 = K. \] (28)

### 1.6.3 Ordinary differential inequality for energy functions

In notations (??) and (??) the inequality (??) takes the form

\[ E(\rho, t) + b(\rho, t) \leq C(J_1 + J_2 + \int_0^t (J_1 + J_2) \, ds). \] (29)

Next we would use the following inequalities

\[ |\partial_t u(x, t)|^\gamma \leq \gamma \int_x^t |\partial_t u|^{\gamma-1} |\partial_x u| \, dx \leq \gamma \left( \int_0^\rho |\partial_t u|^{2-\alpha} \, dx \right)^{\frac{1}{4}} \left( \int_0^\rho |\partial_x u|^2 \, dx \right)^{\frac{3}{4}}, \]

\[ \leq C \left( \int_0^\rho (|\partial_t u|^{2-\alpha} + |\partial_x u|^2) \, dx \right)^{\frac{1}{4}} = C(b)^{\frac{1}{4}}, \] (30)

with \( \gamma = (4 - \alpha)/2 \) and

\[ |\partial_x u(x, t)| \leq \int_0^t |\partial_x u| \, dt \leq t^{\frac{1}{2}} \left( \int_0^t |\partial_x u|^2 \, dt \right)^{\frac{1}{2}} \leq t^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}}. \] (31)
Remind that \( \partial_x u(\rho, 0) = 0 \), if \( \rho \leq \rho_0 \). Applying (??) and (??), we can evaluate \( J_1 \) and \( J_2 \) in the following way

\[
J_1 = |\partial_t u||\partial_x u| \leq Ct^\frac{1}{2} (b)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}},
\]

(32)

\[
J_2 = |\partial_t u||\partial_x u| \leq C (b)^{\frac{1}{2}} (b_\rho)^{\frac{1}{2}}.
\]

(33)

Finally

\[
J = J_1 + J_2 \leq C \left( t^\frac{1}{2} (b)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} + (b)^{\frac{1}{2}} (b_\rho)^{\frac{1}{2}} \right).
\]

(34)

Next using (??) and (??), we obtain the estimates

\[
\int_0^t J_1 ds = \int_0^t |\partial_t u||\partial_x u|ds \leq \left( \int_0^t |\partial_t u|^2 ds \right)^{\frac{1}{2}} \left( \int_0^t |\partial_x u|^2 ds \right)^{\frac{1}{2}} \leq C \left( \int_0^t b^{\frac{2}{3}} ds \right)^{\frac{1}{2}} \left( \int_0^t (E_\rho) ds \right)^{\frac{1}{2}} \leq Ct^\frac{1}{2} \left( \int_0^t b^{\frac{2}{3}} ds \right)^{\frac{1}{2}} \left( \int_0^t (E_\rho) ds \right)^{\frac{1}{2}},
\]

(35)

\[
\int_0^t J_2 ds \leq \left( \int_0^t |\partial_t u|^2 dt \right)^{\frac{1}{2}} \left( \int_0^t |\partial_x u|^2 dt \right)^{\frac{1}{2}} \leq \left( \int_0^t \left( \int_0^t (|\partial_t u|^{2-\alpha} + |\partial_x u|^2) dx \right)^{\frac{1}{2}} dt \right)^{\frac{1}{2}} \left( \int_0^t (b)^{\frac{2}{3}} dt \right)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} \leq \left( \int_0^t (b)^{\frac{2}{3}} dt \right)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}}.
\]

(36)

Joining (??), (??)-(??), we come to

\[
E(\rho, t) + b(\rho, t)
\]

\[
\leq C \left[ t^\frac{1}{2} (b)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} + (b)^{\frac{1}{2}} (b_\rho)^{\frac{1}{2}} + \left( \int_0^t E_\rho ds \right)^{\frac{1}{2}} + (E_\rho)^{\frac{1}{2}} \left( \int_0^t b^{\frac{2}{3}} ds \right)^{\frac{1}{2}} \right].
\]

(37)

Next we will be considering last inequality for \( \epsilon \in [0, t_0] \) with some \( t_0 \in (0, 1) \) which will be chosen later.

Applying the Young inequality with \( \gamma > 1 \), in the following way

\[
\left( t^\frac{1}{2} (b)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} + (b)^{\frac{1}{2}} (b_\rho)^{\frac{1}{2}} \right) \leq \epsilon b + C(\epsilon)(E_\rho + b_\rho)^{\frac{2}{2(2-\alpha)}} , \quad \epsilon \in (0, 1),
\]

\[
\left( \int_0^t (b)^{\frac{2}{3}} dt \right)^{\frac{1}{2}} (E_\rho)^{\frac{1}{2}} \leq C \left( (E_\rho)^{\frac{2}{2(2-\alpha)}} + \left( \int_0^t (b)^{\frac{2}{3}} dt \right)^{\frac{2}{2}} \right).
\]

and choosing \( \epsilon \) in a reasonable way, we reduce (??) to next one

\[
E(\rho, t) + b(\rho, t)
\]

\[
\leq C \left( (E_\rho + b_\rho)^{\frac{2}{2(2-\alpha)}} + t^\frac{1}{2} \left( \int_0^t b^{\frac{2}{3}} ds \right)^{\frac{1}{2}} \left( \int_0^t E_\rho ds \right)^{\frac{1}{2}} + \left( \int_0^t (b)^{\frac{2}{3}} dt \right)^{\frac{1}{2}} \right).
\]

(38)
Assume that 
\[ \sup_{0 \leq t \leq \delta} b(\rho, t) = b(\rho, \delta), \]
\( \delta > 0, \) as for as \( b(\rho, 0) = 0. \) We introduce the function
\[ E(\rho) = \sup_{0 \leq t \leq \delta} E(\rho, t) = \int_0^\delta \int_0^\rho \left( |\partial_t u|^{2-\alpha} + (\partial_x u)^2 \right) dx \, dt, \]
which satisfies
\[ \sup_{0 \leq t \leq \delta} \frac{\partial (E(\rho, t))}{\partial \rho} = \int_0^\delta \left( |\partial_t u|^{2-\alpha} + (\partial_x u)^2 \right) dt = \frac{\partial (E(\rho))}{\partial \rho}. \]
Putting into (??) \( t = \delta, \) we come to
\[ E(\rho) + b(\rho, \delta) \leq C \left( \left( E(\rho) + b(\rho, \delta) \right)^{\frac{\gamma}{\gamma - 1}} + \delta^{\frac{2}{\gamma} b(\rho, \delta)} \right) \left( E(\rho) \right)^{\frac{1}{\gamma}} + \delta^{\frac{2}{\gamma} b(\rho, \delta)}. \]
Applying the Young inequality
\[ \delta^{\frac{2}{\gamma} b(\rho, \delta)} \left( E(\rho) \right)^{\frac{1}{\gamma}} \leq \frac{1}{2} b(\rho, \delta) + C \left( E(\rho) \right)^{\frac{\gamma}{\gamma - 1}}, \]
and choosing \( \delta > 0 \) in a reasonable way, we come finally to ordinary differential inequality
\[ \Lambda^\nu \leq C \Lambda^\rho, \]
with
\[ \Lambda = E(\rho) + b(\rho, \delta), \quad \nu(\alpha) = \frac{2(\gamma - 1)}{\gamma} = \frac{2(2 - \alpha)}{4 - \alpha} < 1 \iff \gamma < 2 \iff 0 < \alpha. \]
Integrating last inequality over \((\rho, \rho_0),\) we come to the desired estimate
\[ \Lambda^{1-\nu}(\rho, \delta) = \left( \sup_{0 \leq s \leq \delta} E(\rho, s) + b(\rho, \delta) \right)^{1-\nu} \leq \Lambda^{1-\nu}(\rho_0, \delta) - (\rho_0 - \rho) \frac{1 - \nu(\alpha)}{C(\alpha, \tau)}. \]

**1.6.4 Finite speed of propagation**

**Proposition 8** Let \( u \) be a weak solution of problem (??), (??) and (??) and condition (??) be fulfilled with \( 0 < \alpha < 1. \) Then there exist positive numbers \( \delta > 0 \) and \( \rho \in (0, \rho_0) \) such that
\[ u(x, t) = 0, \quad 0 \leq x \leq \rho, \quad 0 \leq t \leq \delta, \]
if the global energy \( E(1, T) + b(1, T), \) satisfying to (??), is small.
Proof. According to (??), we have the inequality

$$\Lambda^{1-\nu}(\rho,\delta) \leq (2K)^{1-\nu} - (\rho_0 - \rho)^{1-\nu}. $$

It follows that there exists $\rho > 0$ such that

$$\Lambda^{1-\nu}(\rho,\delta) \leq 0, \quad i.e. \ u(x,t) = 0, \ 0 \leq x \leq \rho, \ 0 \leq t \leq \delta$$

if

$$(2K)^{1-\nu} < \rho_0 \frac{1-\nu}{C}. $$

\[\Box\]

1.6.5 Elliptic equation in semi-discretized degenerated processes

In this section we consider a local weak solution of the problem (see Proposition ??)

$$\int_{\Omega} \left\{ \frac{u-u_0}{h}v + a\left(\frac{u-u_0}{h} + E\right)\nabla[u + \tau \frac{u-u_0}{h}\nabla v] \right\} dx + \int_{\Gamma_s} \beta(\frac{u-u_0}{h})v d\sigma = 0. $$

(40)

Introducing $w = \frac{u-\frac{u_0}{h}}{h}$, we reduce the problem to: find $w$ in $H^1(\Omega)$ such that,

$$\int_{\Omega} \left\{ wv + a(w+E)\nabla[(h+\tau)w+u_0]v\right\} dx + \int_{\Gamma_s} \beta(w)v d\sigma = 0. $$

(41)

For local solutions of equation (??), we can prove the following results:

Assume that

$$\frac{1}{C}|s|^\alpha \leq a(s) \leq C|s|^\alpha, \ 0 < \alpha < 1, C = const > 0,$$

(42)

$$E = 0.$$  (43)

Let us introduce the notation

$$B_\rho(x_0) = \{x \in \Omega : |x - x_0| < \rho\}, \ S_\rho = \partial B_\rho(x_0)$$

$$b(\rho, w) \equiv \int_{B_\rho} (w^2 + (h+\tau)a(w)|\nabla w|^2)dx, \ b_\rho = \int_{S_\rho} w^2 + (h+\tau)a(w)|\nabla w|^2)dx.$$  (44)

Let us consider a local weak solution of equation (??) into the ball $B_{\rho^*}(x_0) \subset \Omega \subset \mathbb{R}^n$ such that

$$b(\rho^*, w) \equiv \int_{B_{\rho^*}} (w^2 + (h+\tau)a(w)|\nabla w|^2)dx \leq K.$$  (45)
**Proposition 9** Let \( u \) be a weak solution to (??) and the conditions (??), (??) are fulfilled. Then:

a) Assume that \( u_0 \equiv 0 \) in \( B_{\rho_0}(x_0) \), \( \rho_0 < \rho^* \). Given an arbitrary point \( x_0 \in \Omega \),
\[
    u(x) \equiv 0 \text{ a.e. in } B_{\rho}(x_0) \tag{46}
\]
with \( \rho \) given by the expression
\[
    \rho \equiv (\rho_0 - CK)_+, \ C = C(\alpha, n). \tag{47}
\]

b) Assume that \( u_0 \equiv 0 \) in \( B_{\rho_0}(x_0) \) satisfying
\[
    \int_{B_\rho} |\nabla u_0|^\gamma dx \leq \varepsilon (\rho - \rho_0)^\mu, \ \rho \in [\rho_0, \rho^*], \tag{48}
\]
with some positive constants \( \gamma, \mu, \varepsilon \). Then there exist positive constants \( K_*, \varepsilon_* \) such that once
\[
    K \leq K_*, \ \varepsilon \leq \varepsilon_*, \tag{49}
\]
any weak solution of equation (??) possesses the property
\[
    u(x) \equiv 0 \text{ a.e. in } B_{\rho_0}(x_0). \tag{50}
\]

**Remark 3** If in (??) \( \rho = 0 \), then the above statement offers no information on the vanishing set of \( w \). Nonetheless, if the total energy \( K \) is small enough, then always \( \rho > 0 \) and the vanishing set of \( w \) is not empty.

### 1.6.6 Finite speed of propagation of solution to equation with \( \tau = 0 \)

We consider special (traveling waves) solutions of equation
\[
    \partial_t u = \partial_x \left[ a(\partial_t u) \partial_x u \right], \ x \in \Omega = ] - 1, 1[ \tag{51}
\]
in the form: for a given positive \( \lambda \),
\[
    u(x, t) = u(\lambda x + t) = u(\xi), \ \xi \geq -1. \tag{52}
\]

Substituting (??) into (??), we obtain the ordinary differential equation
\[
    u' = \lambda^2 (a(u') u')', \tag{53}
\]
or first order differential equation with separated variables
\[
    u = \lambda^2 a(u') u' + C. \tag{54}
\]
If \( a(s) = 0 \) in \( ]-\infty, 0[ \) and \( a(s) > 0 \) elsewhere with \( a(s) = 1 \) if \( s \geq s_0 > 0 \). Then, if one notes \( f(\xi) = \xi a(\xi) \),
\[
    u = \lambda^2 f(u') + C.
\]
\( f^{-1} \) is a graph, \( f^{-1} \) is a function in \([0, \infty[\) and \( f^{-1}(0) = \infty, 0\).

Let us consider \( \beta = f^{-1} \) in \([0, +\infty[\) and \( \beta(s) = 0 \) if \( s \leq 0 \), let us assume that \( \int_0^\infty \frac{dx}{\beta(x)} \) exists and denote by \( B(y) = \int_0^\infty \frac{\lambda^2 dx}{\beta(x)} \).

Since \( \beta(x) > 0 \) for any \( x > 0 \) and has linear growth at infinity, \( B \) is an increasing continuous function on \([C, +\infty[\), derivable in \([C, +\infty[\), with \( \lim_{\infty} B(y) = +\infty \).

Thus, \( B^{-1} \) has got the same properties on \((0, +\infty)\).

Moreover, for any \( \xi > \xi_0 \), \( y'(\xi) := (B^{-1})'(\xi - \xi_0) = \frac{1}{B'(B^{-1}(\xi - \xi_0))} = \beta(\frac{B^{-1}(\xi - \xi_0) - C}{\lambda^2}) = \beta(\frac{y(\xi) - C}{\lambda^2}) \).

\( B^{-1} \) is a continuous function with \( B^{-1}(0) = C \), thus \( (B^{-1})' \) is also a continuous function and
\[
y(\xi_0) = C \quad \text{and} \quad y'(\xi_0) = \beta(\frac{y(\xi_0) - C}{\lambda^2}) = 0.
\]

\( y \) is a global classical solution (see A. F. Filippov [?]) to
\[
y'(y) = \beta(\frac{y - C}{\lambda^2}) \quad \text{in} \quad \mathbb{R} \quad \text{with} \quad y(0) = 0.
\]
different from the trivial solution \( y(\xi) = C \) and satisfying \( y(\xi) = C \) for any \( \xi \leq \xi_0 \).

Traveling-waves solutions.

Consider in the sequel \( C = \xi_0 = 0 \) and \( y \) a non-trivial solution to
\[
y'(y) = \beta(\frac{y}{\lambda^2}) \quad \text{in} \quad \mathbb{R} \quad \text{with} \quad y(0) = 0.
\]

Note that \( y' \geq 0 \) and \( y \geq 0 \) then,
\[
\lambda^2 f(y') = y.
\]

Let us consider \( u(t, x) = y(\lambda x + t) \).

\[
\partial_t u(t, x) = y'(\lambda x + t), \quad \partial_x u = \lambda y'(\lambda x + t)
\]

and \( [a(\partial_t u)\partial_x u](t, x) = \lambda[a(y')y'](t + \lambda x) = \lambda f(y')(t + \lambda x) \),

and then,
\[
\partial_x [a(\partial_t u)\partial_x u](t, x) = \lambda^2[f(y')]'(t + \lambda x) = \partial_t u(t, x).
\]

The initial condition is given by \( u_0(x) = y(\lambda x) \) and the boundary conditions
\[
u(t, -1) = y(t - \lambda) \quad \text{and} \quad u(t, 1) = y(t + \lambda).
\]

At last, note that \( u_0(x) = 0 \) in \([-1, 0]\) and that \( u \) possesses the property of finite speed of propagation (from nonzero disturbances) in the following sense :
\[
u(x, t) = 0, \quad \frac{t}{\lambda} \leq -x < 1.
\]
1.7 Conclusion and open problems

A solution to the problem related to a strictly positive maximum erosion rate $E$ (really the relevant case)

\[
\begin{align*}
\frac{\partial}{\partial t} u - \tau \Delta A(\partial_t u + E) - \text{div}(a(\partial_t u + E)\nabla u) &= 0 \quad \text{in } Q, \\
\frac{\partial}{\partial t} u + E &= 0 \quad \text{in } Q, \\
-\tau \frac{\partial}{\partial n} A(\partial_t u + E) - a(\partial_t u + E)\frac{\partial}{\partial n} u &= 0 \quad \text{in } ]0,T[ \times \Gamma_e, \\
-\tau \frac{\partial}{\partial n} A(\partial_t u + E) - a(\partial_t u + E)\frac{\partial}{\partial n} u &\in \beta(\partial_t u) \quad \text{in } ]0,T[ \times \Gamma_s, \\
{u}_{t=0} &= u_0 \quad \text{in } \Omega.
\end{align*}
\]

has been found in $\text{Lip}([0,T], H^1(\Omega))$.

In order to conclude that this problem is well-posed in the sense of Hadamard, one still has to prove that such a solution is unique. This is still an open problem, mainly due to a behaviour of hysteresis type of the equation.

Let us cite a recent paper of Z. Wang and al. [?] where the uniqueness of the solution to a similar equation has been proved. The equation is posed in the one-dimension space case, and the method is based on a Holmgren approach.

The above solution is a solution to a perturbation of the real problem since $a$ has to be the graph $H$ of the Heaviside function in order to satisfy the initial law of state. This problem is open too, and to our knowledge, the realistic model appears as a non standard free boundary problem for flux limiter $\lambda$ by imposing the original unilateral condition

\[1 - \lambda \geq 0, \quad \frac{\partial}{\partial t} u + E \geq 0, \quad (1 - \lambda) (\partial_t u + E) = 0 \text{ a.e. in } ]0,T[ \times \Omega,\]

via the inclusion $\lambda \in H(\partial_t u + E)$.

References


